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COGRADIENTS IN FUZZY BCK-ALGEBRAS

HEE SIK KIM

ABSTRACT. In this paper we apply the notion of $\rhd_\mu$ and $\lhd_\mu$ to fuzzy BCK-algebra, and show that $\lhd_\mu$ is cogradient to a partial order of the BCK-algebra.

1. Introduction

J. Neggers ([7]) has defined a pogroupoid and he obtained a functorial connection between posets and pogroupoids and associated structure mappings. J. Neggers and H. S. Kim ([8]) demonstrated that a pogroupoid $X(\cdot)$ is modular* if and only if its associated poset $X(\leq)$ is $(C_2 + \underline{1})$-free , a condition which corresponds naturally to the notion of sublattice (in the sense of Kelly-Rival [3, 5]) isomorphic to N_5, and that this is equivalent to the associativity of the pogroupoid. J. Neggers and H. S. Kim ([10]) introduced the notion of the relation $\rhd_\mu$ on fuzzy pogroupoid, and proved that for given a pogroupoid $X(\cdot)$, the associated poset $X(\leq)$ is $(C_2 + \underline{1})$-free iff the relation $\rhd_\mu$ is transitive for any fuzzy subset μ of X. In this paper we apply the notion of $\rhd_\mu$ and $\lhd_\mu$ to fuzzy BCK-algebra, and show that $\lhd_\mu$ is cogradient to a partial order of the BCK-algebra.

2. A relation $\rhd_\mu$

The notion of BCK-algebras was formulated first in 1966 by K. Iséki. This notion was originated from two different ways. One is based on set theory, and the other is propositional calcului. A *BCK-algebra* is a

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non-empty set X together with a binary operation $*$ and a constant 0 satisfying the following axims: for all $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(V) $0 * x = 0$.

The concept of a fuzzy set was introduced by L. A. Zadeh ([16]). A *fuzzy subset* of a set X is a function $\mu : X \to [0,1]$. The applications of fuzzy concepts to posets and groupoids have been investigated by several authors (including [2, 10, 13, 15, 17]). A map $\mu : X \to [0,1]$ is called a *fuzzy subalgebra* of a BCK-algebra X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for any $x, y \in X$. Note that if μ is a fuzzy subalgebra of a BCK-algebra X then $\mu(0) \geq \mu(x)$ for all $x \in X$.

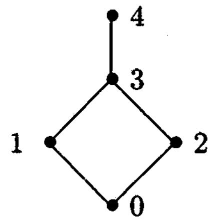
Suppose $(X, *, 0)$ and $(Y, *', 0')$ are two BCK-algebras. A mapping $f : X \to Y$ is called a *BCK-homomorphism* if for any $x, y \in X$, $f(x * y) = f(x) * f(y)$. Moreover, if f is one-one and onto, then we can say f a *BCK-isomorphism* and denote it by $X \cong Y$. With this concept we have the following properties: (i) $f(0) = 0'$, and (ii) if $x * y = 0$ in X, then $f(x) *' f(y) = 0'$ in Y.

On the while, the concept of isomorphism in the poset theory is a little bit different from the concept of BCK-algebras. Even though there is a one-one and onto order-preserving mapping between two posets, the two posets need not be isomorphic ([1]). We say two posets X and Y are *(poset)-isomorphic* if there is a one-one and onto order preserving mapping f and its inverse mapping f^{-1} is also order preserving. There are two ways to define a partially ordered set: (i) *weak inclusion*; reflexive, anti-symmetric, transitive (ii) *strong inclusion*; irreflexive, transitive, and they are equivalent ([12, pp. 1-3]).

In a BCK-algebra X we define a binary operation $\leq$ by $x \leq y$ if and only if $x * y = 0$. We can see that a BCK-algebra contains a poset structure in it. The poset $(X, \leq)$ is said to be the *associated poset* with the BCK-algebra $(X; *, 0)$. The association is not bi-unique, i.e., non-isomorphic BCK-algebras may have order-isomorphic posets associated with them.

EXAMPLE 2.1. Consider the following two BCK-algebras having the same poset structure:

$*_1$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0



$*_2$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Define a map $f : X := \{0, 1, 2, 3, 4\} \to X$ by $f(i) = i$ $(i = 0, 1, 2, 3, 4)$. Then f is a poset isomorphism, but not a BCK-isomorphism, since $f(4 *_1 1) = 4 \neq 3 = f(4) *_2 f(1)$.

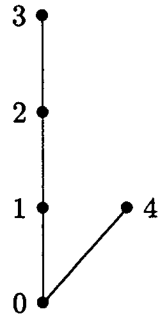
Let $\mu : X \to [0, 1]$ be a fuzzy subset of a BCK-algebra X. Define a relation $\rhd_\mu$ on X by

$$x \rhd_\mu y \iff \mu(x * y) < \mu(y * x).$$

Since $x * x = 0$, $\mu(x * x) < \mu(x * x)$ does not hold, and hence the relation $\rhd_\mu$ is irreflexive. Similarly, we define a relation $\lhd_\mu$ on X by $x \lhd_\mu y \iff \mu(y * x) < \mu(x * y)$.

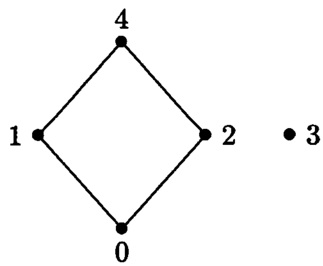
EXAMPLE 2.2. Consider the following BCK-algebra X ([6, pp. 273]).

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0



Define a map $\mu : X \to [0,1]$ by $0 \leq \mu(0) < \mu(3) < \mu(4) < \mu(1) < \mu(2) \leq 1$. Then the transitivity of $\rhd_\mu$ does not hold, since $1 \rhd_\mu 3$ and $3 \rhd_\mu 4$, but not $1 \rhd_\mu 4$. If we define a map $\nu : X \to [0,1]$ by $1 \geq \nu(0) > \nu(4) > \nu(3) > \nu(2) > \nu(1) \geq 0$, then $X(\rhd_\nu)$ is a poset as following left Hasse diagram:

$X(\rhd_\nu)$



$X(\rhd_\xi)$

Moreover, if we define a fuzzy subset $\xi : X \to [0,1]$ on the BCK-algebra $(X, *_1)$ described in Example 2.1 by $0 \leq \xi(0) = \xi(3) < \xi(1) = \xi(2) < \xi(4) \leq 1$, then $X(\rhd_\xi)$ is a poset as the above right Hasse diagram.

THEOREM 2.3. *Let $(X;*,0)$ be a BCK-algebra. Define a fuzzy subset $\mu : X \to [0,1]$ by*

$$\mu(x) := \begin{cases} a & \text{if } x = 0, \\ b & \text{otherwise.} \end{cases}$$

where $0 \leq a < b \leq 1$. Then $X(\rhd_\mu)$ is a poset.

Proof. Let $x \rhd_\mu y$ and $y \rhd_\mu z$. Then $\mu(x*y) < \mu(y*x)$, $\mu(y*z) < \mu(z*y)$. This means $x*y = 0$ and $y*z = 0$, since μ is two-valued. It follows from $X(\leq)$ is a poset that $x \leq z$. By (IV) we obtain $x * z = 0$ and $z * x \neq 0$. Hence $\mu(x * z) = a < b = \mu(z * x)$, i.e., $x \rhd_\mu z$. Thus $X(\rhd_\mu)$ is a poset. □

In Theorem 2.3 we introduced two-valued fuzzy subset μ of a BCK-algebra for $X(\rhd_\mu)$ to be a poset. We pose the following open problem:

PROBLEM. *Under what other condition(s) for $X(\rhd_\mu)$ to be a poset?*

3. Cogradients in fuzzy BCK-algebras

Suppose R_1 and R_2 are relations on a set X. We shall consider relations R_1 and R_2 to be *cogradient* provided that $(x, y) \in R_i$ (or xR_iy) implies $(y, x) \notin R_j$, $i, j = 1, 2$, $i \neq j$, where $x \neq y$. We then obtain the following result.

THEOREM 3.1. *If $(X; *, 0)$ is a BCK-algebra, and if $\mu : X \to [0, 1]$ is a fuzzy subalgebra of this BCK-algebra, then the relations $x \leq y$ iff $x * y = 0$ and $x \lhd_\mu y$ iff $\mu(y * x) < \mu(x * y)$ are cogradient.*

Proof. Let $x, y \in X$ with $x \lhd_\mu y$. If $y < x$, then $x * y \neq 0$, but $y * x = 0$. Hence $\mu(0) = \mu(y * x) < \mu(x * y) \leq \mu(0)$, a contradiction. This means that $y < x$ does not hold. On other hand, let $x \leq y$ in $X(\leq)$. We may assume $x < y$ in $X(\leq)$, since $x \lhd_\mu x$ does not hold. Assume $y \lhd_\mu x$. Then $\mu(x * y) < \mu(y * x)$. Since $x < y$, $x * y = 0$, but $y * x \neq 0$. Hence $\mu(0) = \mu(x * y) < \mu(y * x) \leq \mu(0)$, a contradiction. It follows that $y \lhd_\mu x$ does not hold. This proves the theorem. □

Of course, in the general situation $X(\leq)$ and $X(\rhd_\mu)$ (or $X(\lhd_\mu)$) may fail to be cogradient. A question arises to what extent the cogradience of $X(\leq)$ and $X(\rhd_\mu)$ (or $X(\lhd_\mu)$) influences the "approximate" fuzzy subalgebra structure of the fuzzy subset μ of X.

Suppose that $(X; *, 0)$ is a BCK-algebra and suppose that the fuzzy subset μ is defined as follows:

$$\mu(x) := \begin{cases} a & \text{if } x = 0, \\ b & \text{otherwise.} \end{cases}$$

where $0 \leq a < b \leq 1$. Now suppose $x < y$. Then $x*y = 0$ and $y*x \neq 0$. Hence $\mu(x*y) = a < b = \mu(y*x)$, i.e., $x \rhd_\mu y$. This means that $\rhd_\mu$ is an extension of $<$. Conversely, if $x \rhd_\mu y$, then $\mu(x*y) < \mu(y*x)$, whence $x*y = 0$ and $x < y$, since $y*x \neq 0$. Thus $\rhd_\mu = <$, i.e., $X(<) = X(\rhd_\mu)$ precisely. Thus we summarize:

THEOREM 3.2. *If $(X;*,0)$ is a BCK-algebra and if μ is a fuzzy subset of X where if $x \neq 0$, $\mu(0) = a < b = \mu(x)$, then $X(<) = X(\rhd_\mu)$.*

Thus we may "code" $X(<)$ precisely by taking $a = 0$ and $b = 1$, and within the class $X(\rhd_\mu)$, $X(<)$ will be uniquely determined in this fashion.

Actually, if $(X;*,0)$ is a *d-algebra* ([11]), i.e., if it satisfies conditions (III), (IV) and (V) for the BCK-algebra, then we may use the same scheme, i.e., we set

$$x \lhd_\mu y \quad \text{provided} \quad \mu(y*x) < \mu(x*y).$$

Thus, if $\mu : X \to [0,1]$ is a fuzzy subalgebra of the d-algebra, then $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(0) \geq \mu(x)$ for all $x \in X$.

Suppose now that we define $x < y$ iff $x*y = 0$ in a d-algebra $(X;*,0)$. Then $X(<)$ is not necessarily a poset. However, if μ is a fuzzy subalgebra of X and if $x < y$ then $x*y = 0$ and $y*x \neq 0$, and hence $\mu(y*x) \leq \mu(x*y) = \mu(0)$. It means that either $x \lhd_\mu y$ or $\mu(y*x) = \mu(x*y)$, i.e., $y \lhd_\mu x$ does not hold. Conversely, if $x \lhd_\mu y$, then $y < x$ is impossible. It follows that:

COROLLARY 3.3. Theorem 3.1 *holds if $(X;*,0)$ is a d-algebra.*

Similarly, we obtain:

COROLLARY 3.4. Theorem 3.2 *holds if $(X;*,0)$ is a d-algebra.*

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Hee Sik Kim, Department of Mathematics, Hanyang University, Seoul 133-791, Korea

E-mail: heekim@email.hanyang.ac.kr