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A MASCHKE-TYPE THEOREM FOR THE GRADED SMASH COPRODUCT $C \rtimes kG$

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Abstract. M. Cohen and S. Montgomery showed that a Maschke-type theorem for the smash product, which unlike the corresponding result for group actions, does not require any assumptions about the characteristic of the algebra. Our purpose in this paper is a Maschke-type theorem for the graded smash coproduct $C \rtimes kG$: let V be a right $C \rtimes kG$-comodule and W a $C \rtimes kG$-subcomodule of V which is a C-direct summand of V. Then W is a $C \rtimes kG$-direct summand of V. Also this result is equivalent to the following: let V be a graded right C-comodule and W a graded subcomodule of V which has a complement as a C-subcomodule of V. Then W has a graded complement.

Introduction

M. Cohen and S. Montgomery showed that a grading by G can be considered as a module algebra over the dual algebra kG^*. As coalgebra version of results of their for the smash product $A\#kG^*$, by F. Van Oystaeyen, etc., viewing a G-graded coalgebra over the field k as a right comodule coalgebra over the Hopf algebra kG it is possible to use a Hopf algebraic approach to the study of coalgebras graded by an arbitrary group that was started in [9].

The concepts of smash product and smash coproduct have been proved to be extremely useful tools in the theory of action and coaction of a Hopf algebra on an algebra or on a coalgebra. The mentioned smash product and smash coproduct have been constructed in [2, 3, 7, 8, 9].

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Our purpose in this paper is a Maschke-type theorem for the graded smash coproduct $C \rtimes kG$: let V be a right (left) $C \rtimes kG$-comodule and let W be a $C \rtimes kG$-subcomodule of V which is a C-direct summand of V. Then W is a $C \rtimes kG$-direct summand of V (Theorem 2.1). Also this result is equivalent to the following: let V be a graded (right) C-comodule and W a graded subcomodule of V which has a complement as a C-subcomodule of V. Then W has a graded complement (Theorem 2.2).

1. Preliminaries

Throughout k is a fixed field. All coalgebras and unadorned tensor products $\otimes$ are over k. We refer to [1, 8, 10] for full detail on coalgebras and comodules and further notation and conventions are as in [5, 7, 9]. We denote by $\mathcal{M}^C$ the category of right comodules over the coalgebra C.

Let G be a group with unit element 1. A coalgebra (C, Δ, ε) is G-graded if $C = \bigoplus_{g\in G} C_g$ for k-subspaces C_g such that $\Delta(C_g) \subseteq \sum_{uv=g} C_u \otimes C_v$ for any $g \in G$ and $\varepsilon(C_g) = 0$ for any $g \neq 1$. If $c \in C$ is homogeneous and $\Delta(c) = \sum c_1 \otimes c_2$, we consider everywhere that c_1 and c_2 are also homogeneous. *i.e.*, $\deg c = (\deg c_1)(\deg c_2)$. Graded coalgebras have been studied in [3, 9].

Before we continue we point out an example of graded coalgebras and further examples are as in [9]: If k is a field and G a finite group, then we note by $k[G]$ the free k-module generated by G. If we define the k-linear maps

$$\Delta : k[G] \to k[G] \otimes k[G] \quad \text{and} \quad \varepsilon : k[G] \to k$$

such that $\Delta(g) = \sum_{v\in G} gv^{-1} \otimes v = \sum_{uv=g} u \otimes v$ and $\varepsilon(g) = \delta_{g,1}$ for all $g \in G$ where $\delta_{g,1}$ is the Kronecker symbol, then $(k[G], \Delta, \varepsilon)$ is a G-graded coalgebra with the grading $(k[G])_g = kg$, for all $g \in G$. Here the graded ring $k[G]^*$ associated to the coalgebra $(k[G], \Delta, \varepsilon)$ where the grading is given by

$$(k[G]^*)_g = \{f \in k[G]^* \mid f(C_u) = 0 \text{ for all } u \neq g\}$$

is exactly the group ring kG with the natural grading.

Now, a right C-comodule M with structure map $\rho : M \to M \otimes C$ is graded by G if $M = \bigoplus_{g\in G} M_g$ as k-subspaces such that $\rho(M_g) \subseteq \sum_{uv=g} M_u \otimes C_v$ for all $g \in G$. If M, N are G-graded right C-comodules, then a comodule morphism $f : M \to N$ is said to be a graded morphism if $f(M_g) \subseteq N_g$ for every $g \in G$. In this way we obtain the category of graded right C-comodule, denoted by gr^C.

We know that for any group G the group algebra kG has a Hopf algebra structure determined by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and antipode $\mathcal{S}(g) = g^{-1}$ for any $g \in G$.

The graded smash coproduct $C \rtimes kG$ of the G-graded coalgebra and the Hopf algebra kG is defined as the k-vector space $C \otimes kG$ with

$$\Delta : C \rtimes kG \to (C \rtimes kG) \otimes (C \rtimes kG) \quad \text{and} \quad \varepsilon : C \rtimes kG \to k$$

given by:

$$\Delta(c \rtimes g) = \sum \big(c_1 \rtimes g\mathcal{S}(\deg c_2)\big) \otimes (c_2 \rtimes g) \quad \text{and} \quad \varepsilon(c \rtimes g) = \varepsilon_C(c)$$

for any homogeneous element $c \in C$ and $g \in G$.

We know in [3, 7] that the smash coproduct $C \rtimes kG$ with Δ and ε as above is a coalgebra. We refer to [3, 4, 7] for full detail of smash coproduct in a more general setting and note that the graded smash coproduct appears in a natural way when one studies graded comodules.

A coalgebra (C, Δ, ε) is called a right kG-comodule coalgebra if C is an kG-comodule under $\rho : C \to C \otimes kG$ such that Δ and ε are right kG-comodule maps. Then we have the fact in [3] that a coalgebra C graded by G if and only if C is a right kG-comodule coalgebra.

LEMMA 1.1. *For any finite group G,*

(1) *if a coalgebra C is graded by G. Then the convolution algebra $(C \rtimes kG)^*$ is an algebra isomorphic to the smash product $C^* \# kG^*$.*
(2) *if G acts on the coalgebra D. Then $(D \rtimes kG^*)^*$ is an algebra isomorphic to the skew group algebra $D^* \# kG$.*

2. A Maschke-type theorem for $C \rtimes kG$

We are going to give now the main results of this paper.

M. Cohen and S. Montgomery showed in [2] that a Maschke-type theorem for the smash product, which unlike the corresponding result (cf.

[6]) for group actions, does not require any assumptions about the characteristic of the algebra.

Our purpose in this paper is a coalgebra version of these results:

THEOREM 2.1. *Let V be a right (left) $C \rtimes kG$-comodule and let W be a $C \rtimes kG$-subcomodule of V which is a C-direct summand of V. Then W is a $C \rtimes kG$-direct summand of V.*

Proof. We first consider right comodules. Let $\pi : V \to W$ be the C-comodule splitting homomorphism. So we have $\pi|_W = 1$ and $(\pi \otimes 1_C)\psi_V = \psi_W \pi$ where $\psi_V : V \to V \otimes C$ is a structure map of the C-comodule V and ψ_W the induced structure map of subcomodule W by that of V.

Consider the smash coproduct $C \rtimes kG$, we have the fact that C is a right kG-comodule coalgebra with a structure map $\rho_C : C \to C \otimes kG$. Also by the hypothesis we have V is a $C \rtimes kG$-comodule with a structure map $\varphi_V : V \to V \otimes C \rtimes kG$ via $\varphi_V = (1_V \otimes \rho_C)\psi_V$ as k-linear maps. In fact, we compute

$$\begin{aligned}
(\varphi_V \otimes 1_{C \rtimes kG})\varphi_V &= \big((1_V \otimes \rho_C)\psi_V \otimes 1_{C \rtimes kG}\big)(1_V \otimes \rho_C)\psi_V \\
&= (1_V \otimes \rho_C \otimes \rho_C)(\psi_V \otimes 1_C)\psi_V \\
&= (1_V \otimes \rho_C \otimes \rho_C)(1_V \otimes \Delta_C)\psi_V \\
&= \big(1_V \otimes (\rho_C \otimes \rho_C)\Delta_C\big)\psi_V \\
&= (1_V \otimes \Delta_{C \rtimes kG}\rho_C)\psi_V \\
&= (1_V \otimes \Delta_{C \rtimes kG})(1_V \otimes \rho_C)\psi_V \\
&= (1_V \otimes \Delta_{C \rtimes kG})\varphi_V
\end{aligned}$$

the third equality using that ψ_V is a right C-comodule structure map. Hence $\varphi_V = (1_V \otimes \rho_C)\psi_V$ is a $C \rtimes kG$-comodule structure map of V.

Now, we define $\lambda : V \to W$ by

$$\lambda := (1_W \otimes \varepsilon_{kG})(\pi \otimes 1_{kG})(1_V \otimes \varepsilon_{kG} \otimes 1_{kG})(1_V \otimes u_{kG} \otimes u_{kG}).$$

Then since $\pi|_W = 1$, we have $\lambda|_W = 1$. Also it is enough to show that λ is a $C \rtimes kG$-comodule homomorphism. In fact, we compute

$$\begin{aligned}(\lambda \otimes 1_{C \rtimes kG})\varphi_V &= (\pi \otimes 1_{C \rtimes kG})\varphi_V \\ &= (\pi \otimes 1_{C \rtimes kG})(1_V \otimes \rho_C)\psi_V \\ &= (1_W \otimes \rho_C)(\pi \otimes 1_C)\psi_V \\ &= (1_W \otimes \rho_C)\psi_W \pi \\ &= \varphi_W \pi \\ &= \varphi_W \lambda\end{aligned}$$

the fourth equality using that π is a C-comodule homomorphism. Thus λ is a $C \rtimes kG$-comodule splitting homomorphism.

Similarly, we have a very similar argument works for left comodules. This completes the proof of the theorem. □

We know in [3] that the categories gr^C and $\mathcal{M}^{C \rtimes kG}$ are isomorphic as categories. Thus the above theorem is equivalent to the following:

THEOREM 2.2. *Let V be a graded (right) C-comodule and W a graded subcomodule of V which has a complement as a C-subcomodule of V. Then W has a graded complement.*

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