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ON THE PRUSS EXTENSION OF THE HSU-ROBBINS-ERDŐS THEOREM

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Abstract. The Hsu-Robbins-Erdős theorem states that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables, then $EX_1^2 < \infty$ and $EX_1 = 0$ if and only if $\sum_{n=1}^{\infty} P(|\sum_{k=1}^{n} X_k| \geq n\epsilon) < \infty$ for every $\epsilon > 0$. Under some auxiliary conditions, Spătaru (1994) extended this to the case where the X_n are independent, but their distributions come from a finite set. Pruss (1996) proved Spătaru's result under weaker conditions. The purpose of this paper is to improve Pruss conditions.

1. Introduction

Suppose that $N = \{1, 2, \cdots\}$ is partitioned into p subsets $N_1, \cdots, N_p$, and let $Y_1, \cdots, Y_p$ be p random variables. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that X_n has the same distribution as Y_i whenever $n \in N_i, 1 \leq i \leq p$, and let $S_n = X_1 + \cdots + X_n$. Recently, Spătaru [5] proved that if the auxiliary Conditions **A** and **B**(below) hold, then

(1) $$\sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) < \infty, \quad \forall \epsilon > 0,$$

if and only if

(2) $$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P(|X_k| \geq n) < \infty$$

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and

$$(3) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} EX_k I(|X_k| < n) = 0.$$

Condition **A**. For each $i \in \{1, \cdots, p\}$, there exists $a_i \in [0,1]$ and positive constants $C_1(i), C_2(i)$ such that

$$C_1(i) n^{a_i} \le \alpha_i(n) \le C_2(i) n^{a_i + 1/2}$$

for all n large, where $\alpha_i(t) = \sharp\{l \in N_i : l \le t\}$ for $1 \le i \le p$ and $t \in R$.

Condition **B**. There is a constant C such that , for any $n \in N$ and $i = 1, \cdots, p$,

$$\sum_{k \in [n,\infty) \cap N_i} \frac{1}{k^3} \le C \frac{\alpha_i(n)}{n^3}.$$

For the case of $p = 1$, i.e., $\{X_n, n \ge 1\}$ is a sequence of identically distributed random variables, Hsu-Robbins-Erdős theorem (see, Hsu and Robbins[3] and Erdős [1,2]) states that (1) holds if and only if (2) and (3) hold.

For the general case, Spătaru [5] showed that without the auxiliary conditions (1) always implies both (2) and (3). Pruss [4] and Sung [7] independently proved that the converse is not true. On the other hand, Pruss [4] provided some other weaker auxiliary conditions, for every $i \in \{1, \cdots, p\}$

$$(4) \qquad [(\ \mathbf{V}_i \text{ or } \mathbf{W}_i \text{ or } \mathbf{X}_i) \text{ and } \mathbf{Y}_i] \text{ or } \mathbf{Z}_i,$$

under which (2) and (3) imply (1). The reader should be referred to Pruss' paper to find the definitions of the conditions labeled **V**, **W**, **X** and **Z**. However, Pruss [4] could not prove that (4) can be replaced by $\mathbf{Y}_i$ and left it as an open problem. The Condition $\mathbf{Y}_i$ is as follows.

Condition $\mathbf{Y}_i$. There is a constant C_i such that

$$\alpha_i(2n) \le C_i \alpha_i(n)$$

for n sufficiently large.

Note that Condition **B** implies Condition $\mathbf{Y}_i$ for each $i \in \{1, \cdots, p\}$, and Condition $\mathbf{Z}_i$ implies $\mathbf{Y}_i$ (see, Pruss [4]).

The purpose of this paper is to solve the open problem. In light of the above note, the following theorem improves not only that of Spătaru [5] but also that of Pruss [4].

THEOREM 1. *Assume that Condition $\mathbf{Y}_i$ holds for each $i \in \{1, \cdots, p\}$. Then* (1) *holds if and only if* (2) *and* (3) *hold.*

The proof of the theorem is presented in Section 3. Section 2 collects some preliminary lemmas.

2. Preliminary lemmas

To prove the main theorem we will need the following several lemmas. The next three lemmas come from Pruss [4].

LEMMA 1. *Assume that Condition $\mathbf{Y}_i$ holds for each $i \in \{1, \cdots, p\}$. Then* (2) *holds if and only if*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P(|X_k| \geq n\delta) < \infty \tag{5}$$

for each $\delta > 0$.

LEMMA 2. *Suppose that Y_i has finite mean for each $i \in \{1, \cdots, p\}$. Assume that*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} EX_k = 0. \tag{6}$$

Then the following statements are equivalent:

(i) *condition* (1) *holds, i.e., S_n/n converges completely to 0 as $n \to \infty$.*

(ii) $\sum_{n=1}^{\infty} P(|(S_n^{(i)})^s| \geq n\epsilon) < \infty$, *for every $\epsilon > 0$ and each $i \in \{1, \cdots, p\}$, where $S_n^{(i)} = \sum_{k \in [1,n] \cap N_i} X_k$ and X^s denotes the symmetrized random variable associated with X.*

LEMMA 3. *Suppose that Y_i has finite mean for each $i \in \{1, \cdots, p\}$. Then (3) and (6) are equivalent.*

Now we derive new results under the assumption that Condition $\mathbf{Y}_i$ holds.

LEMMA 4. *Assume that Condition $\mathbf{Y}_i$ holds. Then there exists a constant c such that*

$$cn\alpha_i(n) \le \sum_{k=1}^{n} \alpha_i(k) \le n\alpha_i(n) \tag{7}$$

for n sufficiently large.

Proof. The second inequality of (7) is obvious, since $\alpha_i(\cdot)$ is a non-decreasing function. On the other hand, we get from the monotonicity of $\alpha_i(\cdot)$ and Condition $\mathbf{Y}_i$ that

$$\begin{aligned}\sum_{k=1}^{n} \alpha_i(k) &\ge \left\lfloor \frac{n}{2} \right\rfloor \alpha_i\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \ge \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{C_i} \alpha_i\left(2\left\lfloor \frac{n}{2} \right\rfloor\right) \\ &\ge \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{C_i} \alpha_i(n-1) \ge \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{C_i} (\alpha_i(n) - 1) \ge \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{2C_i} \alpha_i(n)\end{aligned}$$

for n sufficiently large, where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a. Thus the first inequality of (7) is proved. □

By applying Lemma 4, we have the following lemma which plays an essential role in our main theorem.

LEMMA 5. *Assume that Condition $\mathbf{Y}_i$ holds. Then $E|X|\alpha_i(|X|) < \infty$ if and only if $\sum_{n=1}^{\infty} \alpha_i(n) P(|X| \ge n) < \infty$.*

Proof. Noting that

$$\sum_{n=1}^{\infty} \alpha_i(n) P(|X| \ge n) = \sum_{j=1}^{\infty} P(j \le |X| < j+1) \sum_{n=1}^{j} \alpha_i(n),$$

we have by Lemma 4 that

(8)

$$c_1 \sum_{j=1}^{\infty} j\alpha_i(j) P(j \le |X| < j+1) \le \sum_{n=1}^{\infty} \alpha_i(n) P(|X| \ge n) \le \sum_{j=1}^{\infty} j\alpha_i(j) P(j \le |X| < j+1)$$

for some constant c_1. Also, from the following fact

$$E|X|\alpha_i(|X|) = \sum_{j=0}^{\infty} E|X|\alpha_i(|X|) I(j \le |X| < j+1),$$

we get by Condition $\mathbf{Y}_i$ that

$$\sum_{j=1}^{\infty} j\alpha_i(j) P(j \le |X| < j+1) \le E|X|\alpha_i(|X|)$$

(9)

$$\le \sum_{j=0}^{\infty} (j+1)\alpha_i(j+1) P(j \le |X| < j+1) \le c_2 \sum_{j=1}^{\infty} j\alpha_i(j) P(j \le |X| < j+1)$$

for some constant c_2. Thus the conclusion follows from (8) and (9). □

Finally, we will need the following elementary lemma.

LEMMA 6. *Assume that Condition* $\mathbf{Y}_i$ *holds. Then*

$$\alpha_i(2^j t) \le (2C_i)^j \alpha_i(t)$$

for sufficiently large $t \in R$ *and each* $j \in N$.

Proof. From the monotonicity of $\alpha_i(\cdot)$ and Condition $\mathbf{Y}_i$, we have

$$\alpha_i(2t) \le \alpha_i(2(\lfloor t \rfloor + 1)) \le C_i \alpha_i(\lfloor t \rfloor + 1) \le C_i(\alpha_i(t) + 1) \le 2C_i \alpha_i(t)$$

for sufficiently large t. The proof is completed by applying this j-times.□

3. Proof of Theorem 1

The fact that (1) implies (2) and (3) was already proved by Spătaru [5]. Conversely, assume that (2) and (3) hold. The proof is based on certain ideas of Stout [6]. In view of Lemma 1, Lemma 2, and Lemma 3, we can assume that for each n, X_n is symmetric. Let $\epsilon > 0$ be given. For each n and $k(1 \le k \le n)$, define $X'_{nk} = X_k I(|X_k| < n^\rho), X''_{nk} = X_k I(|X_k| \ge n\epsilon/N)$, and $X'''_{nk} = X_k I(n^\rho \le |X_k| < n\epsilon/N)$, where $\rho(0 < \rho < 1)$ and N(positive integer) are chosen later in the proof. Let ${S'_n}^{(i)} = \sum_{k\in[1,n]\cap N_i} X'_{nk}, {S''_n}^{(i)} = \sum_{k\in[1,n]\cap N_i} X''_{nk}$, and ${S'''_n}^{(i)} = \sum_{k\in[1,n]\cap N_i} X'''_{nk}$.

From the inequality $e^x \le 1 + x + \frac{x^2}{2}e^{|x|}$ for all $x \in R$, noting that $|X'_{nk}| < n^\rho$, we have for $t > 0$

$$\begin{aligned} E\left\{\exp\left(\frac{t}{n}X'_{nk}\right)\right\} &\le E\left\{1 + \frac{t}{n}X'_{nk} + \frac{t^2}{2n^2}|X'_{nk}|^2 \exp\left(\frac{t}{n}|X'_{nk}|\right)\right\} \\ &= 1 + E\left\{\frac{t^2}{2n^2}|X'_{nk}|^2 \exp\left(\frac{t}{n}|X'_{nk}|\right)\right\} \\ &\le 1 + \frac{t^2}{2n^2}\exp\left(\frac{t}{n^{1-\rho}}\right) E|X'_{nk}|^2 \\ &\le \exp\left\{\frac{t^2}{2n^2}\exp\left(\frac{t}{n^{1-\rho}}\right) E|X'_{nk}|^2\right\}. \end{aligned}$$

By independence

$$\begin{aligned} E\left\{\exp\left(\frac{t}{n}{S'_n}^{(i)}\right)\right\} &= \prod_{k\in[1,n]\cap N_i} E\left\{\exp\left(\frac{t}{n}X'_{nk}\right)\right\} \\ &\le \exp\left\{\frac{t^2}{2n^2}\alpha_i(n)\exp\left(\frac{t}{n^{1-\rho}}\right) EY_i^2 I(|Y_i| < n^\rho)\right\} \\ &\le \exp\left\{\frac{t^2}{2n^{1-\rho}}\exp\left(\frac{t}{n^{1-\rho}}\right) E|Y_i|\right\}. \end{aligned}$$

Thus, choosing $t = 2\log n/\epsilon$, we have

$$\begin{aligned} P(S_n'^{(i)} \geq n\epsilon) &\leq e^{-t\epsilon} E\left\{\exp\left(\frac{t}{n} S_n'^{(i)}\right)\right\} \\ &\leq \exp\left\{-t\epsilon + \frac{t^2}{2n^{1-\rho}} \exp\left(\frac{t}{n^{1-\rho}}\right) E|Y_i|\right\} \\ &\leq \frac{2}{n^2} \end{aligned}$$

for n sufficiently large, since $\frac{t^2}{n^{1-\rho}} \exp(\frac{t}{n^{1-\rho}}) \to 0$ as $n \to \infty$. Thus

$$\sum_{n=1}^{\infty} P(S_n'^{(i)} \geq n\epsilon) < \infty. \tag{10}$$

From the definition of X''_{nk}, it follows that

$$P(S_n''^{(i)} \geq n\epsilon) \leq \sum_{k \in [1,n] \cap N_i} P\left(|X_k| \geq \frac{n\epsilon}{N}\right).$$

This entails by Lemma 1 that

$$\sum_{n=1}^{\infty} P(S_n''^{(i)} \geq n\epsilon) < \infty. \tag{11}$$

Since $|X'''_{nk}| < n\epsilon/N, S_n'''^{(i)} \geq n\epsilon$ implies that there are at least N non-zero X'''_{nk} for $k \in [1,n] \cap N_i$. Hence

$$\begin{aligned} P(S_n'''^{(i)} \geq n\epsilon) &\leq \sum_{k_1 < \cdots < k_N} P(X'''_{n,k_1} \neq 0, \cdots, X'''_{n,k_N} \neq 0) \\ &\leq \sum_{k_1 < \cdots < k_N} P(|X_{k_1}| \geq n^\rho, \cdots, |X_{k_N}| \geq n^\rho) \\ &= \binom{\alpha_i(n)}{N} [P(|Y_i| \geq n^\rho)]^N \\ &\leq \binom{\alpha_i(n)}{N} [P(|Y_i| \alpha_i(|Y_i|) \geq n^\rho \alpha_i(n^\rho))]^N, \end{aligned}$$

where the summation $\sum_{k_1<\cdots<k_N}$ is taken for all N-tuples $(k_1, \cdots, k_N)$ with $k_1 < \cdots < k_N$ and $k_j \in [1, n] \cap N_i$ for each j. By Markov's inequality,

$$P(S_n'''^{(i)} \geq n\epsilon) \leq \binom{\alpha_i(n)}{N} \left[\frac{E|Y_i|\alpha_i(|Y_i|)}{n^\rho \alpha_i(n^\rho)}\right]^N$$
$$\leq [E|Y_i|\alpha_i(|Y_i|)]^N \frac{\alpha_i^N(n)}{n^{\rho N}\alpha_i^N(n^\rho)}.$$

Thus we have

$$\sum_{n=1}^{\infty} P(S_n'''^{(i)} \geq n\epsilon) \leq [E|Y_i|\alpha_i(|Y_i|)]^N \sum_{n=1}^{\infty} \frac{\alpha_i^N(n)}{n^{\rho N}\alpha_i^N(n^\rho)}$$
$$= [E|Y_i|\alpha_i(|Y_i|)]^N \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{\alpha_i^N(n)}{n^{\rho N}\alpha_i^N(n^\rho)}$$
$$\leq [E|Y_i|\alpha_i(|Y_i|)]^N \sum_{k=0}^{\infty} \frac{\alpha_i^N(2^{k+1})}{2^{k(\rho N-1)}\alpha_i^N(2^{k\rho})}$$
$$\leq c[E|Y_i|\alpha_i(|Y_i|)]^N \sum_{k=0}^{\infty} \frac{(2C_i)^{N(k(1-\rho)+2)}}{2^{k(\rho N-1)}},$$

for some constant $c > 0$. The last inequality follows from Lemma 6, since

$$\alpha_i(2^{k+1}) \leq \alpha_i(2^{\lfloor k(1-\rho)\rfloor+2}2^{k\rho})$$
$$\leq (2C_i)^{\lfloor k(1-\rho)\rfloor+2}\alpha_i(2^{k\rho})$$
$$\leq (2C_i)^{k(1-\rho)+2}\alpha_i(2^{k\rho})$$

for k sufficiently large. Choose $0 < \rho < 1$ such that $(2C_i)^{1-\rho}/2^\rho < 1$ and then choose large integer N such that $2((2C_i)^{1-\rho}/2^\rho)^N < 1$. Then we have, noting that $E|Y_i|\alpha_i(|Y_i|) < \infty$ by Lemma 5, that

$$(12) \qquad \sum_{n=1}^{\infty} P(S_n'''^{(i)} \geq n\epsilon) < \infty.$$

From (10), (11), and (12), we get that

$$\sum_{n=1}^{\infty} P(S_n \geq 3p\epsilon n) < \infty, \tag{13}$$

since

$$\sum_{n=1}^{\infty} P(S_n \geq 3p\epsilon n)$$
$$\leq \sum_{n=1}^{\infty} \sum_{i=1}^{p} [P({S'_n}^{(i)} \geq n\epsilon) + P({S''_n}^{(i)} \geq n\epsilon) + P({S'''_n}^{(i)} \geq n\epsilon)].$$

By replacing X_k by $-X_k$ in (13), we obtain that

$$\sum_{n=1}^{\infty} P(S_n \leq -3p\epsilon n) < \infty. \tag{14}$$

Combining (13) and (14) gives

$$\sum_{n=1}^{\infty} P(|S_n| \geq 3p\epsilon n) < \infty,$$

and so (1) is proved.

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