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ON A FUNCTIONAL EQUATION ASSOCIATED WITH STOCHASTIC DISTANCE MEASURES

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Abstract. The general solution of the functional equation $f_1(pr, qs) + f_2(ps, qr) = g(p, q) + h(r, s)$ for $p, q, r, s \in]0, 1[$ will be investigated without any regularity assumptions on the unknown functions $f_1, f_2, g, h :]0, 1[\to \mathbb{R}$.

1. Introduction

Let I denote the open unit interval $]0, 1[$. Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_1 = \{x \in \mathbb{R}_+ | x \neq 1\}$. Let

$$\Gamma_n^o = \left\{ P = (p_1, p_2, ..., p_n) \,|\, 0 < p_k < 1, \sum_{k=1}^{n} p_k = 1 \right\}$$

denote the set of all n-ary discrete complete probability distributions (without zero probabilities), that is Γ_n^o is the class of discrete distributions on a finite set Ω of cardinality n with $n \geq 2$. Over the years, many distance measures between discrete probability distributions have been proposed.

Almost all similarity, affinity or distance measures $\mu_n : \Gamma_n^o \times \Gamma_n^o \to \mathbb{R}_+$ that have been proposed between two discrete probability distributions can be represented in the *sum form*

$$\mu_n(P, Q) = \sum_{k=1}^{n} \phi(p_k, q_k), \tag{1}$$

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where $\phi : I \times I \rightarrow \mathbb{R}$ is a real-valued function on unit square, or a monotonic transformation of the right side of (1), that is,

$$\mu_n(P,Q) = \psi\left(\sum_{k=1}^{n} \phi(p_k, q_k)\right), \tag{2}$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ is a monotone function on $\mathbb{R}$. The function ϕ is called a *generating function.* It is also referred to as the *kernel* of $\mu_n(P,Q)$. Some important examples of sum form distance measures between two discrete probability distributions P and Q in Γ_n^o are (see [2]):

(a) *Directed divergence*

$$\phi(x,y) = x(\log x - \log y),$$

$$D_n(P,Q) = \sum_{k=1}^{n} p_k \log\left(\frac{p_k}{q_k}\right);$$

(b) *Symmetric J-divergence*

$$\phi(x,y) = (x-y)(\log x - \log y),$$

$$J_n(P,Q) = \sum_{k=1}^{n} (p_k - q_k)\log\left(\frac{p_k}{q_k}\right);$$

(c) *Hellinger coefficient*

$$\phi(x,y) = \sqrt{xy},$$

$$H_n(P,Q) = \sum_{k=1}^{n} \sqrt{p_k q_k};$$

(d) *Jeffreys distance*

$$\phi(x,y) = \left(\sqrt{x} - \sqrt{y}\right)^2,$$

$$K_n(P,Q) = \sum_{k=1}^{n} \left(\sqrt{p_k} - \sqrt{q_k}\right)^2;$$

(e) *Chernoff coefficient*

$$\phi(x,y) = x^{\alpha} y^{1-\alpha},$$

$$C_{n,\alpha}(P,Q) = \sum_{k=1}^{n} p_k^{\alpha} q_k^{1-\alpha} \qquad \alpha \in I;$$

(f) *Variational distance*

$$\phi(x,y) = |x-y|,$$

$$V_n(P,Q) = \sum_{k=1}^{n} |p_k - q_k|;$$

(g) *Proportional distance*

$$\phi(x,y) = \min\{x,y\},$$

$$X_n(P,Q) = \sum_{k=1}^{n} \min\{p_k, q_k\};$$

(h) *Kagan affinity measure*

$$\phi(x,y) = \frac{(y-x)^2}{y},$$

$$A_n(P,Q) = \sum_{k=1}^{n} q_k \left[1 - \frac{p_k}{q_k}\right]^2;$$

(i) *Vajda affinity measure*

$$\phi(x,y) = y\left|\frac{x}{y} - 1\right|^{\alpha},$$

$$A_{n,\alpha}(P,Q) = \sum_{k=1}^{n} q_k \left|\frac{p_k}{q_k} - 1\right|^{\alpha}, \qquad \alpha \geq 1;$$

(j) *Matusita distance*

$$\phi(x,y) = |x^{\alpha} - y^{\alpha}|^{\frac{1}{\alpha}},$$

$$M_{n,\alpha}(P,Q) = \sum_{k=1}^{n} |p_k^{\alpha} - q_k^{\alpha}|^{\frac{1}{\alpha}}, \qquad 0 < \alpha \leq 1;$$

(k) *Divergence measure of degree* α

$$\phi(x,y) = \frac{1}{2^{\alpha-1}-1}\left[x^{\alpha}y^{1-\alpha} - x\right],$$

$$B_{n,\alpha}(P,Q) = \frac{1}{2^{\alpha-1}-1}\left[\sum_{k=1}^{n}\left(p_k^{\alpha}q_k^{1-\alpha} - p_k\right)\right], \qquad \alpha \neq 1;$$

(l) *Cosine α-divergence measure*

$$\phi(x,y)=\frac{1}{2}\left[x-\sqrt{xy}\cos\left(\alpha\log\left(\frac{x}{y}\right)\right)\right],$$

$$N_{n,\alpha}(P,Q)=\frac{1}{2}\left[1-\sum_{k=1}^{n}\sqrt{p_kq_k}\cos\left(\alpha\log\frac{p_k}{q_k}\right)\right];$$

(m) *Divergence measure of Higashi and Klir*

$$\phi(x,y)=x\log\frac{2x}{x+y}+y\log\frac{2y}{x+y},$$

$$I_n(P,Q)=\sum_{k=1}^{n}\left[p_k\log\left(\frac{2p_k}{p_k+q_k}\right)+q_k\log\left(\frac{2q_k}{p_k+q_k}\right)\right];$$

(n) *Csiszar f-divergence measure*

$$\phi(x,y)=xf\left(\frac{x}{y}\right),$$

$$Z_{n,\alpha}(P,Q)=\sum_{k=1}^{n}p_kf\left(\frac{p_k}{q_k}\right);$$

(o) *Kullback-Leibler type f-distance measure*

$$\phi(x,y)=x[f(x)-f(y)],$$

$$L_{n,\alpha}(P,Q)=\sum_{k=1}^{n}p_k[f(p_k)-f(q_k)].$$

The following,
(p) *Renyi's divergence measure*

$$\phi(x,y)=x^{\alpha}y^{1-\alpha}\qquad\qquad \psi(x)=\frac{1}{\alpha-1}\log x$$

$$R_{n,\alpha}(P||Q)=\frac{1}{\alpha-1}\log\left(\sum_{k=1}^{n}p_k^{\alpha}q_k^{1-\alpha}\right),\qquad \alpha\neq 1,$$

is a monotonic transformation of sum form distance measures. Renyi's divergence measure is the logarithm of the so-called exponential entropy

$$E_{n,\alpha}(P,Q) = \left(\sum_{k=1}^{n} p_k^{\alpha} q_k^{1-\alpha}\right)^{\frac{1}{\alpha-1}}.$$

In order to derive axiomatically the principle of minimum divergence, Shore and Johnson [7] formulated a set of four axioms namely *uniqueness, invariance, system independence* and *subset independence.* They proved that if a functional $\mu_n : \Gamma_n^o \times \Gamma_n^o \to \mathbb{R}_+$ satisfies the axioms of uniqueness, invariance and subset independence, then there exists a generating function (or kernel) $\phi : I \times I \to \mathbb{R}$ such that

$$\mu_n(P,Q) = \sum_{k=1}^{n} \phi(p_k, q_k)$$

for all $P, Q \in \Gamma_n^o$. In view of this result one can conclude that the above sum form representation is not artificial. In most applications involving distance measures between probability distributions one encounters the minimization of $\mu_n(P,Q)$ and the sum form representation makes problems tractable.

A sequence of measures $\{\mu_n\}$ is said to be *symmetrically compositive* if for some $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mu_{nm}(P\star R, Q\star S) &+ \mu_{nm}(P\star S, Q\star R)\\ &= 2\mu_n(P,Q) + 2\mu_m(R,S) + \lambda\mu_n(P,Q)\mu_m(R,S)\end{aligned}$$

for all $P, Q \in \Gamma_n^o$, $S, R \in \Gamma_m^o$. If $\lambda = 0$, then $\{\mu_n\}$ is said to be *symmetrically additive.*

To characterize symmetrically compositive sumform distance measures with a measurable generating function, one encounters the following functional equation (see [6]):

$$f(pr,qs) + f(ps,qr) = g(p,q)f(r,s) + g(r,s)f(p,q), \quad (p,q,r,s \in]0,1[),$$

where f and g are real-valued functions on the square of the open unit interval $]0,1[$. The authors of [6] have determined the general solution of the above functional equation when p, q, r, s are in the open-closed unit interval $]0,1]$ since their method relies on the end point 1. Without the boundary point 1, the technique proposed in [6] for solving the above equation does not work. To determine the general solution of the above functional equation one needs the solution of the following functional equation:

$$f(pr, qs) + f(ps, qr) = 2f(p, q) + 2f(r, s), \tag{3}$$

for all $p, q, r, s \in]0, 1[$. For functional equations related to characterization of stochastic distance measures the interested readers should refer to [1-6].

The aim of this paper is to determine the general solution of the following functional equation

$$f_1(pr, qs) + f_2(ps, qr) = g(p, q) + h(r, s), \tag{4}$$

for all $p, q, r, s \in]0, 1[$, which includes the functional equation (3) as a special case. We are interested in solving equation (4) since it can be used in characterization of inset information measures and inset stochastic distance measures.

2. Notation and terminology

A map $L : \mathbb{R}_+ \to \mathbb{R}$ is called *logarithmic* if and only if $L(xy) = L(x) + L(y)$ for all $x, y \in \mathbb{R}_+$. A function $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ is called *bilogarithmic* if and only if it is logarithmic in each variable. The capital letter L along with its subscripts is used exclusively for logarithmic map.

3. Some auxiliary results

The following auxiliary results are needed to determine the general solution of the functional equation (4).

LEMMA 3.1. *The function $f : I^2 \to \mathbb{R}$ satisfies the functional equation*

$$f(pr, qs) = f(p, q) + f(r, s) \tag{5}$$

for all $p, q, r, s \in I$ if and only if

$$f(p, q) = L_1(p) + L_2(q), \tag{6}$$

where $L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic.

Proof. Let $a \in I$ be a fixed element and consider

$$\begin{aligned} f(p,q) &= f(p,q) + 2f(a,a) - 2f(a,a) \\ &= f(paa, qaa) - 2f(a,a) \\ &= f(pa \cdot a, a \cdot qa) - 2f(a,a) \\ &= f(pa,a) + f(a,qa) - 2f(a,a) \\ &= L_1(p) + L_2(q), \end{aligned}$$

where

$$\begin{aligned} L_1(p) &= f(pa,a) - f(a,a) \\ L_2(q) &= f(a,qa) - f(a,a). \end{aligned}$$

Next, we show that L_1 and L_2 are logarithmic functions on $\mathbb{R}_+$. Observe that

$$\begin{aligned} L_1(pq) &= f(pqa,a) - f(a,a) \\ &= f(pqa,a) + f(a,a) - 2f(a,a) \\ &= f(pqaa, aa) - 2f(a,a) \\ &= f(pa \cdot qa, a \cdot a) - 2f(a,a) \\ &= f(pa,a) + f(qa,a) - 2f(a,a) \\ &= L_1(p) + L_1(q) \end{aligned}$$

for all $p, q \in I$. Hence L_1 is logarithmic. It is well known that L_1 can be extended to $\mathbb{R}_+$ from I. Similarly, it can be shown that L_2 is a logarithmic map on $\mathbb{R}_+$. This completes the proof of the lemma. □

LEMMA 3.2. *The functions $f, g, h : I^2 \to \mathbb{R}$ satisfy the functional equation*

$$f(pr, qs) = g(p,q) + h(r,s) \tag{7}$$

for all $p, q, r, s \in I$ if and only if

$$f(p,q) = L_1(p) + L_2(q) + \alpha + \beta, \tag{8}$$

$$g(p,q) = L_1(p) + L_2(q) + \alpha, \tag{9}$$

$$h(r,s) = L_1(r) + L_2(s) + \beta, \tag{10}$$

where $L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic and α, β are arbitrary constants.

Proof. Let a and b be any two fixed elements in I. Inserting $r = a$ and $s = b$ in (7), we obtain

$$g(p,q) = f(pa,qb) - h(a,b). \tag{11}$$

Now again letting $p = b$ and $q = a$ in (7), we have

$$h(r,s) = f(rb,sa) - g(b,a). \tag{12}$$

By (11) and (12), (7) yields

$$f(pr,qs) = f(pa,qb) + f(rb,sa) + k, \tag{13}$$

where $k = -h(a,b) - g(b,a)$. Replacing p by bp, q by aq, r by ar, and s by bs in (13), we obtain

$$f(abpr,abqs) = f(abp,abq) + f(abr,abs) + k. \tag{14}$$

Defining

$$F(p,q) = f(abp,abq) + k \tag{15}$$

we see that the last equation transforms into

$$F(pr,qs) = F(p,q) + F(r,s) \tag{16}$$

for all $p, q, r, s \in I$. By Lemma 3.1, we obtain

$$F(p,q) = L_1(p) + L_2(q), \tag{17}$$

where $L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic. Therefore

$$f(abp,abq) = L_1(p) + L_2(q) - k \tag{18}$$

which is

$$f(p,q) = L_1(p) + L_2(q) + \gamma \tag{19}$$

where γ is a constant. Using (19) in (11) and (12), we obtained (9) and (10), respectively. Letting (19), (9) and (10) into (7), we see that $\gamma = \alpha + \beta$ and thus we have (8). This completes the proof. □

LEMMA 3.3. *The function $f : I^2 \to \mathbb{R}$ satisfies the functional equation*

$$f(pr,qs) + f(ps,qr) = 2f(p,q) + 2f(r,s) \tag{20}$$

for all $p, q, r, s \in I$ if and only if

$$f(p,q) = L(p) + L(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right), \tag{21}$$

where $L : \mathbb{R}_+ \to \mathbb{R}$ is logarithmic and $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ is bilogarithmic.

Proof. Setting $r = \lambda = s$ in (20), we obtain

$$f(\lambda p, \lambda q) = f(p, q) + L(\lambda), \tag{22}$$

where

$$L(\lambda) := f(\lambda, \lambda). \tag{23}$$

It is easy to show that L is logarithmic. Consider

$$f(\lambda_1\lambda_2 p, \lambda_1\lambda_2 q) = f(p, q) + L(\lambda_1\lambda_2). \tag{24}$$

Also we get

$$f(\lambda_1\lambda_2 p, \lambda_1\lambda_2 q) = f(\lambda_2 p, \lambda_2 q) + L(\lambda_1) = f(p, q) + L(\lambda_2) + L(\lambda_1). \tag{25}$$

Thus from (24) and (25), we see that $L(\lambda_1\lambda_2) = L(\lambda_1) + L(\lambda_2)$ for all $\lambda_1, \lambda_2 \in I$. Hence L is logarithmic on I and it can be extended uniquely to $\mathbb{R}_+$.

Now we extend f to $\bar{f}$ from I^2 to $\mathbb{R}_+^2$ as follows: For $p, q \in \mathbb{R}_+$, choose $\lambda \in \mathbb{R}_+$ sufficiently small such that $\lambda, \lambda p, \lambda q \in I$. Define

$$\bar{f}(p, q) = f(\lambda p, \lambda q) - L(\lambda). \tag{26}$$

It is easy to show that $\bar{f}$ in (26) is well defined, that is independent of the choice of λ. To show this, using (22) we write $f(\lambda\mu p, \lambda\mu q)$ in two different ways:

$$f(\lambda\mu p, \lambda\mu q) = f(\lambda p, \lambda q) + L(\mu)$$

and also

$$f(\lambda\mu p, \lambda\mu q) = f(\mu p, \mu q) + L(\lambda).$$

Hence from the last two equations, we get

$$f(\lambda p, \lambda q) - L(\lambda) = f(\mu p, \mu q) - L(\mu). \tag{27}$$

Thus $\bar{f}$ is independent of the choice of λ.

Next we establish that $\bar{f}$ satisfies the functional equation (20). Choose $p, q, r, s \in \mathbb{R}_+$ and $\lambda \in I$ such that $\lambda p, \lambda q, \lambda r, \lambda s \in I$. Next we compute

$$\begin{aligned}\bar{f}(pr, qs) + \bar{f}(ps, qr) &= f(\lambda^2 pr, \lambda^2 qs) + f(\lambda^2 ps, \lambda^2 qr) - 2L(\lambda^2)\\ &= f(\lambda p\lambda r, \lambda q\lambda s) + f(\lambda p\lambda s, \lambda q\lambda r) - 4L(\lambda)\\ &= 2f(\lambda p, \lambda q) + 2f(\lambda r, \lambda s) - 4L(\lambda)\\ &= 2\bar{f}(p, q) + 2\bar{f}(r, s).\end{aligned}$$

Hence $\bar{f}$ satisfies (20) for all $p, q, r, s \in \mathbb{R}_+$. Here after, we simply assume that f satisfies (20) for all $p, q, r, s \in \mathbb{R}_+$.

A substitution of $p = q = r = s = 1$ in (20) yields $f(1, 1) = 0$. Further, substituting $p = q = 1$ in (20), we see that $f(r, s) = f(s, r)$ for all $r, s \in \mathbb{R}_+$. Letting $q = s = 1$ in (20), we have

$$f(p, r) = 2g(p) + 2g(r) - g(pr) \tag{28}$$

where

$$g(p) := f(p, 1). \tag{29}$$

Note that $g(1) = 0$ in view of $f(1, 1) = 0$. Letting $s = 1$ in (20) and then using (28) in the resulting equation, we obtain

$$g(pqr) + g(p) + g(q) + g(r) = g(pr) + g(qr) + g(pq) \tag{30}$$

for $p, q, r \in \mathbb{R}_1$. Defining

$$2\ell(p, r) = g(pr) - g(p) - g(r) \tag{31}$$

we see that (30) reduces to

$$\ell(pq, r) = \ell(p, r) + \ell(q, r) \tag{32}$$

for all $p, q, r \in \mathbb{R}_1$. Hence ℓ is logarithmic on $\mathbb{R}_1^2$ on the first variable. Since the right side of (31) is symmetric with respect to p and r, so also the left side. Thus

$$\ell(p, r) = \ell(r, p),$$

that is ℓ is a real-valued bilogarithmic function on $\mathbb{R}_1^2$.

Again, defining

$$G(p) = g(p) - \ell(p, p) \tag{33}$$

and using (33) in (31) and the symmetry of ℓ, we obtain

$$G(pr) = G(p) + G(r). \tag{34}$$

Thus

$$G(p) = L(p), \qquad p \in \mathbb{R}_1, \tag{35}$$

where $L : \mathbb{R}_1 \to \mathbb{R}$ is an arbitrary logarithmic function. Now using (35) and (33), we obtain

$$g(p) = L(p) + \ell(p, p) \tag{36}$$

for all $p \in \mathbb{R}_1$. The equation (36) in (28) yields

$$f(p,q) = L(p) + L(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right) \tag{37}$$

for all $p, q \in \mathbb{R}_1$. By (29) and (36), we see that

$$f(p,1) = L(p) + \ell(p,p)$$

for all $p \in \mathbb{R}_1$, and since $f(1,1) = 0$, this extends to all $\mathbb{R}_+$. Using the fact that $f(p,1) = L(p) + \ell(p,p)$ for all $p \in \mathbb{R}_+$, and the symmetry of f, we see that (37) holds for all $p, q \in \mathbb{R}_+$. This completes the proof. □

LEMMA 3.4. *The function $f : I^2 \to \mathbb{R}$ satisfies the functional equation*

$$f(pr, qs) + f(ps, qr) = 2f(p,q) + f(r,s) + f(s,r) \tag{38}$$

for all $p, q, r, s \in I$ if and only if

$$f(p,q) = L_0(q) - L_0(p) + L_1(p) + L_1(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right), \tag{39}$$

where $L_0, L_1 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic and $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ is bilogarithmic.

Proof. As in the proof of the previous lemma, we define $\bar{f} : \mathbb{R}_+^2 \to \mathbb{R}$ as

$$\bar{f}(p,q) = f(\lambda p, \lambda q) - L(\lambda),$$

where $L(\lambda) := f(\lambda, \lambda)$. Then L is logarithmic and $\bar{f}$ satisfies the functional equation (38) for all $p, q, r, s \in \mathbb{R}_+$. Hence from here on, we simply assume that f satisfies (38) for all $p, q, r, s \in \mathbb{R}_+$.

Interchanging p with r and q with s in (38), we get

$$f(pr, qs) + f(qr, ps) = 2f(r,s) + f(p,q) + f(q,p). \tag{40}$$

Subtracting (40) from (38), we obtain

$$\phi(ps, qr) = \phi(p,q) + \phi(s,r), \tag{41}$$

where

$$\phi(p,q) := f(p,q) - f(q,p). \tag{42}$$

From Lemma 3.1, we have $\phi(p,q) = 2L_2(p) + 2L_0(q)$, where $L_0, L_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic. Since ϕ is skew symmetric, we see that $L_2 = -L_0$ and hence

$$f(q,p) = f(p,q) + 2L_0(p) - 2L_0(q). \tag{43}$$

Letting (43) into (38), we see that

$$f(pr, qs) + f(ps, qr) = 2f(p, q) + 2f(r, s) + 2[L_0(r) - L_0(s)]. \tag{44}$$

Defining

$$F(p, q) = f(p, q) + [L_0(p) - L_0(q)], \tag{45}$$

we obtain from (44)

$$F(pr, qs) + F(ps, qr) = 2F(p, q) + 2F(r, s)$$

for all $p, q, r, s \in \mathbb{R}_+$. Hence by Lemma 3.3, we have

$$F(p, q) = L_1(p) + L_1(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right), \tag{46}$$

where $L_1 : \mathbb{R}_+ \to \mathbb{R}$ is logarithmic and $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ is bilogarithmic. From (45) and (46), we get the asserted solution (39). This completes the proof. □

LEMMA 3.5. *The functions $f, g, h : I^2 \to \mathbb{R}$ satisfy the functional equation*

$$f(pr, qs) + f(ps, qr) = g(p, q) + h(r, s) \tag{47}$$

for all $p, q, r, s \in I$ if and only if

$$f(p, q) = L_0(q) - L_0(p) + L_1(p) + L_1(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right) + \alpha + \beta, \tag{48}$$

$$g(p, q) = 2\left[L_0(q) - L_0(p) + L_1(p) + L_1(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right)\right] + 2\alpha, \tag{49}$$

$$h(r, s) = 2\left[L_1(r) + L_1(s) + \ell\left(\frac{r}{s}, \frac{r}{s}\right)\right] + 2\beta, \tag{50}$$

where $L_0, L_1 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic and $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ is bilogarithmic, and α, β are arbitrary real constants.

Proof. Let $a \in I$ be a fixed element. Substituting $r = a = s$ in (47), we get

$$g(p, q) = 2f(pa, qa) - h(a, a). \tag{51}$$

Similarly, letting $p = a = q$ in (47), we have

$$h(r, s) = f(ra, sa) + f(sa, ra) - g(a, a). \tag{52}$$

Letting (51) and (52) into (47), we obtain

(53) $f(pr,qs)+f(ps,qr)=2f(pa,qa)+f(ra,sa)+f(sa,ra)+2\alpha_0.$

Replacing p by pa, r by ra, q by qa, and s by sa, we obtain

$$F(pr,qs)+F(ps,qr)=2F(p,q)+F(r,s)+F(s,r), \tag{54}$$

where

$$F(p,q):=f\left(pa^2,qa^2\right)+\alpha_0. \tag{55}$$

By Lemma 3.4 and (55), we obtain

$$f(p,q)=L_0(q)-L_0(p)+L_1(p)+L_1(q)+\ell\left(\frac{p}{q},\frac{p}{q}\right)+\gamma, \tag{56}$$

where γ is a constant. From (56) and (51), we obtain (49). Similarly, (56) and (52), we obtain (50). Letting (49), (50) and (56) into (47), we have $\gamma=\alpha+\beta$. Hence, with this and (56), we have the asserted solution (48). □

LEMMA 3.6. *The functions $f,g:I^2\to\mathbb{R}$ satisfy the functional equation*

$$f(pr,qs)-f(ps,qr)=g(r,s) \tag{57}$$

for all $p,q,r,s\in I$ if and only if

$$f(p,q)=\phi(pq)+L(q), \tag{58}$$

$$g(p,q)=L(q)-L(p), \tag{59}$$

where $L:\mathbb{R}_+\to\mathbb{R}$ is logarithmic and $\phi:\mathbb{R}_+\to\mathbb{R}$ is arbitrary.

Proof. First, we substitute $r=a=q$ in (57), we obtain

$$f(pa,sa)-f(ps,a^2)=g(a,s). \tag{60}$$

Replacing p by pa and s by sa in (60) and rearranging, we get

$$f(a^2p,a^2s)=f(a^2ps,a^2)+g(a,as). \tag{61}$$

Similarly, replacing p by pa, r by ra, s by sa, and q by qa in (57), we have

$$f(a^2pr,a^2qs)-f(a^2ps,a^2qr)=g(ar,as). \tag{62}$$

Using (61) in (62), we obtain

$$g(a,aqs)-g(a,aqr)=g(ar,as). \tag{63}$$

Again letting $p = a = q$ in (57), we get

(64) $$f(ar, as) - f(as, ar) = g(r, s).$$

Hence, by (60), the equation (64) yields

(65) $$g(r, s) = g(a, s) - g(a, r).$$

As before, by replacing r by ar and s by as, we have

(66) $$g(ar, as) = g(a, as) - g(a, ar).$$

Thus from (66) and (65), we see that

(67) $$g(a, aqs) - g(a, aqr) = g(a, as) - g(a, ar).$$

Defining $\psi : I^2 \longrightarrow \mathbb{R}$ as

(68) $$\psi(x) = g(a, ax)$$

we have from (67)

(69) $$\psi(qs) - \psi(qr) = \psi(s) - \psi(r)$$

for all $q, s, r \in I$. From (69), we obtain

(70) $$\psi(qs) - \psi(s) = \psi(qr) - \psi(r)$$

for all $q, s, r \in I$. The last equation yields

(71) $$\psi(qs) - \psi(s) = \delta(q),$$

where $\delta : I \longrightarrow \mathbb{R}$. Interchanging q with s in (71), we see that

(72) $$\psi(sq) - \psi(q) = \delta(s).$$

From (71) and (72), we have

$$\delta(q) - \psi(q) = \delta(s) - \psi(s) = k,$$

where k is a constant. Hence

(73) $$\delta(q) = \psi(q) + k.$$

Using (73) in (72), we have

(74) $$\psi(sq) - \psi(q) + \psi(s) + k.$$

Thus

(75) $$\psi(x) = L(x) + k,$$

where $L : \mathbb{R}_+ \to \mathbb{R}$ is logarithmic. From (68), (65) and (75), we obtain

$$g(r, s) = L(s) - L(r). \tag{76}$$

Using (60) and (76), we see that

$$f(pa, sa) = \sigma(ps) + g(a, s), \tag{77}$$

where

$$\sigma(ps) := f(ps, a^2). \tag{78}$$

Now (76) and (76) give

$$f(pa, sa) = \sigma(ps) + L(s) + \alpha, \tag{79}$$

where α is a constant. Letting (79) into (57) we observe that σ is an arbitrary function. Hence

$$f(p, s) = \phi(ps) + L(s), \tag{80}$$

where $\phi = \sigma + \alpha$. This completes the proof of the lemma. □

4. The main result

Now we are ready to determine the general solution of (4) without any regularity assumptions on the unknown functions.

THEOREM 4.1. *The functions $f_1, f_2, g, h : I^2 \to \mathbb{R}$ satisfy the functional equation*

$$f_1(pr, qs) + f_2(ps, qr) = g(p, q) + h(r, s), \tag{81}$$

for all $p, q, r, s \in I$ if and only if

$$f_1(p, q) = \mathcal{L}(p, q) + L_2(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right) + \phi(pq) - \alpha, \tag{82}$$

$$f_2(p, q) = \mathcal{L}(p, q) - L_2(q) + \ell\left(\frac{p}{q}, \frac{p}{q}\right) - \phi(pq) - \alpha, \tag{83}$$

$$g(p, q) = 2\mathcal{L}(p, q) + 2\ell\left(\frac{p}{q}, \frac{p}{q}\right) + 2\beta - 2\alpha, \tag{84}$$

$$h(r, s) = 2L_1(r) + 2L_1(s) + L_2(s) - L_2(r) + 2\ell\left(\frac{r}{s}, \frac{r}{s}\right) - 2\beta, \tag{85}$$

where $\mathcal{L}(p,q) := L_0(q) - L_0(p) + L_1(p) + L_1(q)$, $L_0, L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}$ *are logarithmic,* $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ *is bilogarithmic,* $\phi : \mathbb{R}_+ \to \mathbb{R}$ *is arbitrary, and* α, β *are arbitrary real constants.*

Proof. It is easy to check that solution (82)-(85) enumerated in the theorem satisfies (81).

Interchanging r with s in (81), we get

$$f_1(ps, qr) + f_2(pr, qs) = g(p,q) + h(s,r). \tag{86}$$

Adding (86) to (81), we obtain

$$F(pr, qs) + F(ps, qr) = 2g(p,q) + 2H(r,s), \tag{87}$$

where

$$F(p,q) = f_1(p,q) + f_2(p,q) \tag{88}$$

$$2H(r,s) = h(r,s) + h(s,r). \tag{89}$$

Further, subtracting (86) from (81), we obtain

$$f(pr, qs) - f(ps, qr) = k(r,s), \tag{90}$$

where

$$f(p,q) = f_1(p,q) - f_2(p,q) \tag{91}$$

$$2H(r,s) = h(r,s) - h(s,r). \tag{92}$$

Using (88), (89), (91), and (92), we obtain

$$f_1 = \frac{1}{2}(F + f), \qquad f_2 = \frac{1}{2}(F - f), \qquad h = \frac{1}{2}(2H + k). \tag{93}$$

From Lemma 3.6, Lemma 3.5, and (93) the asserted solution follows. This completes the proof of the theorem. □

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