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THE WEAK LAW OF LARGE NUMBERS FOR RANDOMLY WEIGHTED PARTIAL SUMS

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ABSTRACT. In this paper we establish the weak law of large numbers for randomly weighted partial sums of random variables and study conditions imposed on the triangular array of random weights $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ and on the triangular array of random variables $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ which ensure that $\sum_{j=1}^{n} W_{nj}|X_{nj} - B_{nj}|$ converges in probability to 0, where $\{B_{nj} : 1 \le j \le n,\ n \ge 1\}$ is a centering array of constants or random variables.

1. Introduction

Let $\{X_n : n \ge 1\}$ be a sequence of independent random variables and $\{w_{nj} : 1 \le j \le n,\ n \ge 1\}$ a triangular array of numbers. The strong laws of large numbers for the weighted partial sums $\sum_{j=1}^{n} w_{nj}X_j$ are extensively investigated in the literature (see, for example, Stout (1968), Chow and Lai (1973), Teicher (1985), Yu (1990) and Cuzick (1995) among others). Let $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ be a triangular array of random variables. The convergence in probability of weighted partial sums of the form $\sum_{j=1}^{n} w_{nj}X_{nj}$ to zero has been studied by several authors (see, for example, Jamison, Orey and Pruitt (1965), Pruitt (1966) and Adler, Rosalsky and Taylor (1991) among others) and the weak laws of large numbers have been applied to the bootstrap (see, for example, Athreya (1983), Csörgö (1992) and Arenal-Gutierrez, Matran and Cuesta-Albertas (1996)).

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In this paper, we study the convergence in probability of $\sum_{j=1}^{n} W_{nj}|X_{nj} -B_{nj}|$ to 0, where $\{B_{nj} : 1 \le j \le n,\ n \ge 1\}$ is a triangular array of numbers (or random variables).

In Section 2 we derive conditions on $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ and on $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ which ensure that the weak law of large number of the form

$$\sum_{j=1}^{n} W_{nj}\ |X_{nj} - B_{nj}|\ \xrightarrow{P}\ 0 \tag{1}$$

holds for the case where EW_{nj}'s are constants and in Section 3 we also construct general conditions which ensure that the weak law of large numbers (1) holds.

2. Preliminaries

LEMMA 2.1. *Let $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ be a triangular array of random variables with $E|W_{nj}| < \infty$. Assume that there exists a triangular array of constants $\{a_{nj} : 1 \le j \le n,\ n \ge 1,\ a_{nj} \neq 0\}$ such that for all $\epsilon > 0$*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|W_{nj} - EW_{nj}| \ge \epsilon|a_{nj}|\} < \infty. \tag{2}$$

Then

$$\max_{1 \le j \le n} |W_{nj} - EW_{nj}|\ /\ |a_{nj}|\ \xrightarrow{P}\ 0. \tag{3}$$

Proof. By Boole's inequality for probability measure (2) implies for all $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{\cup_{j=1}^{n}\ [\ |W_{nj} - EW_{nj}| \ge \epsilon|a_{nj}|\ ]\ \}\ <\ \infty \tag{4}$$

and by the Borel-Cantelli lemma (4) implies for all $\epsilon > 0$

$$P\{\cup_{j=1}^{n}\ [\ |W_{nj} - EW_{nj}| \ge \epsilon|a_{nj}\ |\ ]\ i.\ o.\ \}\ =\ 0. \tag{5}$$

It follows from (5) that we obtain

$$P\left(\lim\ \sup_{n\to\infty} \left\{\max_{1\le j\le n}\left|\frac{W_{nj} - EW_{nj}}{a_{nj}}\right| \ge \epsilon\ \right\}\right)\ =\ 0. \tag{6}$$

Hence, the desired result follows. □

LEMMA 2.2. *Let* $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ *be a triangular array of random variables with* $E|W_{nj}| < \infty$. *Assume that there exists a triangular array of constants* $\{a_{nj} : 1 \le j \le n,\ n \ge 1,\ a_{nj} \ne 0\}$ *such that for all* $\epsilon > 0$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{Var(W_{nj})}{a_{nj}^2} < \infty. \tag{7}$$

Then (3) *holds.*

Proof. By Chebyshev's inequality, (7) implies (2). Hence, the result follows by Lemma 2.1. □

THEOREM 2.1. *Let* $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ *be a triangular array of random variables and let* $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ *be a triangular array of integrable random weights satisfying the condition that there exists a sequence of constants* $\{a_n : n \ge 1,\ a_n \ne 0\}$ *such that for all* $\epsilon > 0$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} P\{\ |W_{nj} - EW_{nj}| \ \ge\ \epsilon\ |a_n|\ \} < \infty. \tag{8}$$

Assume

$$EW_{nj} = a_n \text{ for all } 1 \le j \le n,\ n \ge 1 \tag{9}$$

and

$$a_n \sum_{j=1}^{n} |X_{nj} - B_{nj}| \xrightarrow{P} 0. \tag{10}$$

Then (1) *holds.*

Proof. In Lemma 2.1 by putting $a_{nj} = a_n$ for $1 \le j \le n,\ n \ge 1$ it follows from (8) that

$$a_n^{-1} \max_{1\le j\le n} |W_{nj} - EW_{nj}| \xrightarrow{P} 0. \tag{11}$$

First we observe the following equality:

$$\begin{aligned}(12) \qquad & \sum_{j=1}^{n} W_{nj}|X_{nj} - B_{nj}| \\ &= \sum_{j=1}^{n} (W_{nj} - EW_{nj})\ |X_{nj} - B_{nj}| + \sum_{j=1}^{n}\ EW_{nj}\ |X_{nj} - B_{nj}| \\ &= \sum_{j=1}^{n} (W_{nj} - EW_{nj})\ |X_{nj} - B_{nj}| + a_n \sum_{j=1}^{n}\ |X_{nj} - B_{nj}|.\end{aligned}$$

By (10) the second term on the right-hand side of (12) converges in probability to 0. It remains to show that the first term on the right-hand side of (12) converges in probability to 0. To see this, we consider the following:

$$\begin{aligned}(13) \qquad & \left|\sum_{j=1}^{n}(W_{nj} - EW_{nj})|X_{nj} - B_{nj}|\right| \\ &\leq \sum_{j=1}^{n} |W_{nj} - EW_{nj}||X_{nj} - B_{nj}| \\ &\leq \max_{1\leq j\leq n} |W_{nj} - EW_{nj}| \sum_{j=1}^{n} |X_{nj} - B_{nj}| \\ &= (a_n^{-1} \max_{1\leq j\leq n} |W_{nj} - EW_{nj}|) \left(a_n \sum_{j=1}^{n} |X_{nj} - B_{nj}|\right)\end{aligned}$$

Hence, it follows from (10) and (11) that the right-hand side of (13) also converges in probability to 0 and the proof of Theorem 2.1 is complete. □

From Lemma 2.2 and Theorem 2.1 we obtain the following corollary:

COROLLARY 2.1. *Let $\{X_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ be a triangular array of random variables and let $\{W_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ be a triangular array of integrable random weights with the condition that there exists a triangular array of constants $\{a_n : n \geq 1,\ a_n \neq 0\}$ satisfying (9) and*

(10). *If*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{Var(W_{nj})}{a_n^2}<\infty \tag{14}$$

then (1) *holds.*

THEOREM 2.2. *Let* $\{X_{nj}:1\le j\le n,\ n\ge 1\}$ *be a triangular array of random variables and let* $\{W_{nj}:1\le j\le n,\ n\ge 1\}$ *be a triangular array of integrable random weights. Let* $\{a_n, n\ge 1,\ a_n\neq 0\}$ *be a sequence of constants satisfying* (9) *and* (10). *If*

$$a_n^{-1}\max_{1\le j\le n}|W_{nj}-EW_{nj}|\xrightarrow{P}0 \tag{15}$$

then (1) *holds.*

Proof.

$$\begin{aligned}\sum_{j=1}^{n}W_{nj}|X_{nj}-B_{nj}| &= \sum_{j=1}^{n}(W_{nj}-EW_{nj})|X_{nj}-B_{nj}|+\sum_{j=1}^{n}(EW_{nj})|X_{nj}-B_{nj}|\\ &= \sum_{j=1}^{n}(W_{nj}-EW_{nj})|X_{nj}-B_{nj}|+a_n\sum_{j=1}^{n}|X_{nj}-B_{nj}|.\end{aligned} \tag{16}$$

The second term on the right-hand side of (16) converges in probability to 0 by (10). It remains to show that the first term on the right-hand side of (16) converges in probability to 0. It is clear that

$$\begin{aligned}&\left|\sum_{j=1}^{n}(W_{nj}-EW_{nj})|X_{nj}-B_{nj}|\right|\\ &\le (a_n^{-1}\max|W_{nj}-EW_{nj}|)\left(a_n\sum_{j=1}^{n}|X_{nj}-B_{nj}|\right)\xrightarrow{P}0\end{aligned}$$

by (10) and (15). Hence the desired result follows. □

Now we extend Theorem 2.2 to the triangular array of constants $\{a_{nj}: 1\le j\le n,\ n\ge 1\}$.

THEOREM 2.3. *Let* $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ *and* $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ *be triangular arrays of random variables. Assume that there exists a triangular array of constants* $\{a_{nj} : 1 \le j \le n,\ n \ge 1,\ a_{nj} \ne 0\}$ *satisfying the conditions that for all* $\epsilon > 0$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} P\{\ |W_{nj} - EW_{nj}| \ \ge\ \epsilon |a_{nj}|\ \} \ <\ \infty, \tag{17}$$

$$EW_{nj} \ =\ a_{nj}, \tag{18}$$

and

$$\sum_{j=1}^{n} \ |a_{nj}|\ |X_{nj} - B_{nj}| \ \xrightarrow{P}\ 0. \tag{19}$$

Then (1) *holds.*

Proof. According to Lemma 2.1 it follows from (17) that

$$\max_{1 \le j \le n} \ |W_{nj} - EW_{nj}|\ /\ |a_{nj}| \ \xrightarrow{P}\ 0. \tag{20}$$

Now, we consider the following equality:

$$\begin{aligned}
&\sum_{j=1}^{n} W_{nj}\ |X_{nj} - B_{nj}| \\
&= \sum_{j=1}^{n} (W_{nj} - EW_{nj})|X_{nj} - B_{nj}| + \sum_{j=1}^{n} (EW_{nj}|X_{nj} - B_{nj}|) \\
&= \sum_{j=1}^{n} (W_{nj} - EW_{nj})|X_{nj} - B_{nj}| + \sum_{j=1}^{n} (a_{nj}|X_{nj} - B_{nj}|\ ).
\end{aligned} \tag{21}$$

It follows from (19) that the second term on the right-hand side of (21) converges in probability to 0. It remains to show that the first term on the right-hand side of (21) converges in probability to 0. Note that

$$(22)\quad \left|\sum_{j=1}^{n}(W_{nj}-EW_{nj})|X_{nj}-B_{nj}|\right|$$
$$=\sum_{j=1}^{n}(a_{nj})^{-1}|W_{nj}-EW_{nj}||X_{nj}-B_{nj}|(a_{nj})$$
$$\leq \max_{1\leq j\leq n}[\ |W_{nj}-EW_{nj}|\ /\ |a_{nj}|\ ]\sum_{j=1}^{n}[\ |a_{nj}||X_{nj}-B_{nj}|\ ]\ \xrightarrow{P}\ 0$$

by (19) and (20). Hence the desired result follows. □

THEOREM 2.4. *Let* $\{X_{nj}: 1\leq j\leq n,\ n\geq 1\}$ *and* $\{W_{nj}: 1\leq j\leq n,\ n\geq 1\}$ *be triangular arrays of random variables with* $E|W_{nj}| < \infty$. *If* (18), (19) *and* (20) *are satisfied then* (1) *holds.*

COROLLAY 2.2. *Let* $\{X_{nj}: 1\leq j\leq n,\ n\geq 1\}$ *and* $\{W_{nj}: 1\leq j\leq n,\ n\geq 1\}$ *be triangular arrays of random variables satisfying* (18) *and* (19). *If*

$$(23)\qquad \max_{1\leq j\leq n}|W_{nj}-EW_{nj}\ |\ /\ |EW_{nj}\ \xrightarrow{P}\ 0$$

then (1) *holds.*

3. Main results

DEFINITION 3.1. (Lehmann, 1983) A sequence of random variables $\{Y_n\}$ is said to be bounded in probability if for every $\epsilon > 0$ there exist positive numbers M and n_0 such that

$$P(\ |Y_n|\ >\ M\ )\ <\ \epsilon \text{ for all } n\ >\ n_0.$$

In Theorem 2.2, in order that (1) holds, the expression (15) converges in probability not necessarily to 0, i.e., if $a_n^{-1}(\max_{1\leq j\leq n}|W_{nj}-EW_{nj}|\ )$ is bounded in probability then (1) still holds. From this fact we obtain the following theorem.

THEOREM 3.1. *Let* $\{X_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *and* $\{W_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *be triangular arrays of random variables. Let* $\{a_n : n \geq 1,\ a_n \neq 0\}$ *be a sequence of constants satisfying* (10). *If*

$$(24) \qquad a_n^{-1} \max_{1\leq j\leq n} |W_{nj} - a_n| \text{ is bounded in probability.}$$

Then (1) *holds.*

Proof. First we observe the following equality:

$$(25) \qquad \sum_{j=1}^{n} W_{nj}\ |X_{nj} - B_{nj}| = \sum_{j=1}^{n} (W_{nj} - a_n)|X_{nj} - B_{nj}| + a_n \sum_{j=1}^{n} |X_{nj} - B_{nj}|$$

the second term on the right-hand side of (25) converges in probability to 0 by (10). It remains to show that the first term on the right-hand side of (25) converges in probability to 0. To show this we consider

$$\begin{aligned} \left|\sum_{j=1}^{n}(W_{nj} - a_n)|X_{nj} - B_{nj}|\right| &\leq \sum_{j=1}^{n} |W_{nj} - a_n||X_{nj} - B_{nj}| \\ &\leq \max_{1\leq j\leq n} |W_{nj} - a_n| \sum_{j=1}^{n} |X_{nj} - B_{nj}| \\ &= (a_n^{-1} \max_{1\leq j\leq n} |W_{nj} - a_n|) \\ &\quad \cdot \left(a_n \sum_{j=1}^{n} |X_{nj} - B_{nj}|\right) \xrightarrow{P} 0 \end{aligned}$$

by (10) and (24). Thus the proof is complete. □

Next, by deriving sufficient conditions for (10) to hold we consider the weak law of large numbers for randomly weighted partial sums of a triangular array of random variables.

THEOREM 3.2. *Let* $\{W_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *and* $\{X_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *be triangular arrays of random variables and let* $\{a_n : n \geq 1,\ a_n \neq 0\}$ *be a sequence of numbers satisfying* (24) *and*

$$(26) \qquad n|a_n| \ \rightarrow\ 0.$$

Assume

(27) $$\max_{1\le j\le n} |X_{nj} - B_{nj}| \text{ is bounded in probability.}$$

Then (1) holds.

Proof. It is enough to show that (10) holds according Theorem 3.1. Note that

$$\frac{1}{n}\sum_{j=1}^{n} |X_{nj} - B_{nj}| \le \max_{1\le j\le n} |X_{nj} - B_{nj}|.$$

Hence,

$$\begin{aligned} a_n \sum_{j=1}^{n} |X_{nj} - B_{nj}| &= (na_n)\left(\frac{1}{n}\sum_{j=1}^{n} |X_{nj} - B_{nj}|\right) \\ &\le (n|a_n|) \max_{1\le j\le n} |X_{nj} - B_{nj}| \xrightarrow{P} 0 \end{aligned}$$

by (26) and (27). Thus the proof is complete by Theorem 3.1. □

COROLLAY 3.1. *Let $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ and $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ be triangular arrays of random variables and let $\{a_n : n \ge 1,\ a_n \ne 0\}$ be a sequence of numbers satisfying (24) and $na_n = O(1)$. If*

(28) $$\max_{1\le j\le n} |X_{nj} - B_{nj}| \xrightarrow{P} 0$$

then (1) holds.

Proof. The proof is similar to that of Theorem 3.2. □

In Theorem 2.4 in order that (1) holds, the expression (20) converges in probability not necessarily to 0, that is, under assumption that $\max_{1\le j\le n} |W_{nj} - EW_{nj}| \,/\, |a_{nj}|$ is bounded in probability (1) still holds.

From this fact we obtain the following result.

THEOREM 3.3. *Let $\{X_{nj} : 1 \le j \le n,\ n \ge 1\}$ and $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ be triangular arrays of random variables. Assume that there exists a triangular array of constants $\{a_{nj} : 1 \le j \le n,\ n \ge 1\}$ such that*

(29) $$\max_{1\le j\le n} |a_{nj}|^{-1} |W_{nj} - a_{nj}| \text{ is bounded in probability.}$$

If

$$\sum_{j=1}^{n} |a_{nj}|\ |X_{nj} - B_{nj}\ | \xrightarrow{P} 0 \tag{30}$$

then (1) *holds.*

Proof.

$$\begin{aligned} &\sum_{j=1}^{n} W_{nj}|X_{nj} - B_{nj}| \\ &= \sum_{j=1}^{n}(W_{nj} - a_{nj})|X_{nj} - B_{nj}| + \sum_{j=1}^{n} a_{nj}|X_{nj} - B_{nj}| \end{aligned} \tag{31}$$

The second term on the right-hand side of (31) converges in probability to 0 by (30). It remains to show that the first term on the right-hand side of (31) converges in probability to 0. Note that

$$\begin{aligned} &\sum_{j=1}^{n} |W_{nj} - a_{nj}||X_{nj} - B_{nj}| \\ &= \sum_{j=1}^{n}(|a_{nj}|^{-1})(|W_{nj} - a_{nj}||a_{nj}||X_{nj} - B_{nj}|) \\ &\leq \max_{1\leq j\leq n}(|a_{nj}|^{-1}\ |W_{nj} - a_{nj}|) \sum_{j=1}^{n}(\ |a_{nj}||X_{nj} - B_{nj})|\ ) \xrightarrow{P} 0 \end{aligned} \tag{32}$$

by (29) and (30). Thus the desired result follows. □

Finally, corresponding to Theorems 3.1, 3.2 and 3.3 we obtain similar results respectively:

THEOREM 3.4. *Let* $\{X_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *and* $\{W_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *be triangular arrays of random variables satisfying the condition that there exists a sequence of constants* $\{a_n : n \geq 1,\ a_n \neq 0\}$ *such that*

(33) $a_n^{-1} \max_{1\leq j\leq n} |W_{nj} - a_n|$ *is bounded with probability* 1.

Assume

$$(34) \qquad |a_n| \sum_{j=1}^{n} |X_{nj} - B_{nj}| \xrightarrow{P} 0.$$

Then (1) holds.

Proof. First, it follows from (33) that $\{a_n^{-1} W_{nj}\}$ is bounded with probability 1. Hence,

$$\begin{aligned} \left| \sum_{j=1}^{n} W_{nj} |X_{nj} - B_{nj}| \right| &= \left| \sum_{j=1}^{n} a_n^{-1} W_{nj} \, a_n |X_{nj} - B_{nj}| \right| \\ &\leq C \, |a_n| \sum_{j=1}^{n} |X_{nj} - B_{nj}| \xrightarrow{P} 0 \end{aligned}$$

by (34) where C is a constant. □

THEOREM 3.5. *Let $\{W_{nj} : 1 \leq j \leq n, \ n \geq 1\}$ and $\{X_{nj} : 1 \leq j \leq n, \ n \geq 1\}$ be triangular arrays of random variables and $\{a_n : n \geq 1, \ a_n \neq 0\}$ be a sequence of numbers satisfying (26) and (33). Assume*

$$(35) \qquad \max_{1 \leq j \leq n} |X_{nj} - B_{nj}| \text{ is bounded with probability } 1.$$

Then $\sum_{j=1}^{n} W_{nj} \, |X_{nj} - B_{nj}|$ converges to 0 with probability 1.

Proof. It follows from (34) that $\{a_n^{-1} W_{nj}\}$ is bounded with probability 1. Hence

$$\begin{aligned} \left| \sum_{j=1}^{n} W_{nj} |X_{nj} - B_{nj}| \right| &= \left| \sum_{j=1}^{n} a_n^{-1} W_{nj} a_n |X_{nj} - B_{nj}| \right| \\ &\leq C \max_{1 \leq j \leq n} |X_{nj} - B_{nj}| \sum_{j=1}^{n} |a_n| \\ &\leq C \max_{1 \leq j \leq n} |X_{nj} - B_{nj}| \, n |a_n| \rightarrow 0 \end{aligned}$$

with probability 1 and the result follows. □

THEOREM 3.6. *Let $\{X_{nj} : 1 \leq j \leq n, \ n \geq 1\}$ and $\{W_{nj} : 1 \leq j \leq n, \ n \geq 1\}$ be triangular arrays of random variables. Assume that there* *exists a triangular array of constants* $\{a_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ *satisfying* (30). *Assume*

(36) $\max_{1 \leq j \leq n} |a_{nj}|^{-1}(W_{nj} - a_{nj})$ *is bounded with probability* 1

Then (1) *holds.*

Proof. It follows from (36) that $\{a_{nj}^{-1} W_{nj}\}$ is bounded with probability 1. Hence

$$\left|\sum_{j=1}^{n} W_{nj}\ |X_{nj} - B_{nj}|\right| \leq \sum_{j=1}^{n} |a_{nj}^{-1} W_{nj} a_{nj}\ |X_{nj} - B_{nj}||$$

$$\leq C \sum_{j=1}^{n} |a_{nj}||X_{nj} - B_{nj}| \xrightarrow{P} 0$$

by (30) and the proof is complete. □

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