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AVERAGE DISTANCES AND OCTAHEDRAL NORMS

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ABSTRACT. In [6], Godefroy defined octahedral norms to give an isomorphic characterization of spaces containing ℓ_1. Here we will show that such norms can be defined by using "average distances", as introduced in [1]. Also, we indicate some other properties of average distances: in particular, we give some estimates for their values in the product of two spaces, furnished with the max or the sum norm.

1. Introduction and notation

Let (X, $\|.\|$) be a Banach space, of dimension at least two, over the real field R.

We shall use the following notations:

$S_X = \{x \in X;\ \|x\| = 1\}$; we shall simply write S instead of S_X when no confusion can arise;

X^* will denote the dual of X;

$\mathsf{F}(S) = \{F \subset S;\ F \text{ is finite and nonempty}\}$.

If $F = \{x_1, x_2, \cdots, x_n\} \subset S$ and $x \in X$, we set

$$\mu(F, x) = \frac{1}{n} \sum_{i=1}^{n} \|x_i - x\|.$$

We do not exclude, when we write $F = \{x_1, x_2, \cdots, x_n\}$, that $x_i = x_j$ for some pairs i, j.

For $F \in \mathsf{F}(S)$, we also set

$\mu(F, S) = \{\alpha \geq 0;\ \text{there exists } x \in S \text{ such that } \mu(F, x) = \alpha\}$;

$\mu_1(F) = \inf\{\mu(F, S)\}$;

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$\mu_2(F) = \sup\{\mu(F,S)\}$.
Given $F \in \mathsf{F}(S)$, since S is connected, $\mu(F,S)$ is an interval; so $\overline{\mu(\mathsf{F},S)} = [\mu_1(F), \mu_2(F)]$. Now set

$$\mu_1(X) = \sup\{\mu_1(F);\ F \in \mathsf{F}(S)\}$$

and

$$\mu_2(X) = \inf\{\mu_2(F);\ F \in \mathsf{F}(S)\}.$$

We recall the following result (See [10], p. 332):

LEMMA 1.1. *For any Banach space X we have*

$$\mu_1(X) \leq \mu_2(X). \tag{1.1}$$

For any $F \in \mathsf{F}(S)$ (and any X) we have:

$$\max\{1, \mu_1(F)\} \leq \mu_1(X) \leq \mu_2(X) \leq \mu_2(F) \leq 2. \tag{1.2}$$

Note that $[\mu_1(X), \mu_2(X)] = \bigcap\{\overline{\mu(F,S)};\ F \in \mathsf{F}(S)\}$.
If $\mu \in [\mu_1(X), \mu_2(X)]$, then μ is called an *average distance* for X.

For general results on average distances, see [1]-[3] and [7]-[9].
In sections 2 and 3 of this paper we study the condition $\mu_2(X) = 2$, which is strictly connected with the property of containing a copy of ℓ_1; in sections 4 and 5 we study how μ_1 and μ_2 behave when we consider the product of two spaces: we indicate results concerning the "extreme cases", of products performed with the max or the sum norm.

2. A characterization of octahedral norms

The following definition was introduced in [6].

DEFINITION 1. We say that a norm in X is *octahedral* if for every finite dimensional subspace F of X and a every $\eta > 0$, there exists $y \in S_X$ such that for every $x \in F$, we have

$$\|x + y\| \geq (1 - \eta) \cdot (\|x\| + 1). \tag{2.1}$$

Recall the following result (see [4], Theorem III.2.5).

PROPOSITION 2.1. *For any Banach space X the following are equivalent:*

(a) *X contains a subspace isomorphic to ℓ_1;*

(b) *there exists an octahedral norm on X.*

Next theorem is the main result of this section.

THEOREM 2.1. *For a normed space X, the following are equivalent:*

(c) *the norm of X is octahedral;*

(d) $\mu_2(X) = 2$.

Proof. It is clear that if X has an octahedral norm, then for every finite subset Φ of S_X we have $\mu_2(\Phi) = 2$, thus $\mu_2(X) = 2$: so (c) implies (d).

Now we prove that (d) implies (c). Let $\mu_2(X) = 2$. Take a finite dimensional subspace F of X and let $\eta \in (0,1)$; if y is an arbitrary point of S_X, then for any $x \in X$ we have:

$$\frac{\|x+y\|}{\|x\|+1} \geq \frac{|\|x\|-1|}{\|x\|+1};$$

if $\|x\| \geq 1$, then the last term is $= 1 - \frac{2}{\|x\|+1}$, and so $\geq 1-\eta$ when $\|x\| \geq \frac{2}{\eta} - 1$; if $\|x\| \leq 1$, then the last term is $= \frac{2}{\|x\|+1} - 1$, and so it is $\geq 1-\eta$ when $\|x\| \leq \frac{\eta}{2-\eta} = 1 - \frac{2-2\eta}{2-\eta}$. Now set $C = \left\{x \in F; \frac{\eta}{2-\eta} \leq \|x\| \leq \frac{2}{\eta} - 1\right\}$: C is a compact subset of F, so we can find, for any $\varepsilon \in (0,1)$, a finite ε-net for it, say G. Let G contain n elements, say $g_1, g_2, \cdots, g_n$; set $g_i' = \frac{g_i}{\|g_i\|}$ (note that $\Theta \notin G$ if ε is small enough).

Take $y \in S_X$ such that $\frac{1}{n}\sum_{i=1}^{n} \|y - g_i'\| \geq 2 - \frac{\varepsilon}{n}$, thus $\|y - g_i'\| \geq 2n - \varepsilon - 2(n-1) = 2 - \varepsilon$ for every i.

For any $x \in C$, we can choose a point in G, say g, such that $\|x + g\| < \varepsilon$: let $g = tg'$ with $\|g'\| = 1$ and some $t \geq 0$. We obtain:

$$\frac{\|y+x\|}{\|x\|+1} \geq \frac{\|y-g\|-\varepsilon}{1+\|g\|+\varepsilon}.$$

Then consider $f(\tau) = \|y - \tau g'\|$; note that $f(0) = 1$ and $f(1) \geq 2 - \varepsilon$; if $t \leq 1$, by using the fact that $f(\tau)$ is 1-Lipschitz, we obtain: $f(t) \geq 2 - \varepsilon - (1-t)$, which implies

$$\frac{\|y-g\|-\varepsilon}{1+\|g\|+\varepsilon} \geq \frac{1-\varepsilon+t-\varepsilon}{1+t+\varepsilon} = 1 - \frac{3\varepsilon}{1+t+\varepsilon}.$$

Since the last function increases with t $(0 \le t \le 1)$, we obtain

$$\frac{\|y-g\|-\varepsilon}{1+\|g\|+\varepsilon} \ge 1 - \frac{3\varepsilon}{1+\varepsilon}.$$

If $t \ge 1$, by using the fact that $f(\tau)$ is convex we obtain:
$2-\varepsilon \le \|y-g'\| = f(1) \le (1-\frac{1}{t})\cdot f(0) + \frac{1}{t}\cdot f(t) = (1-\frac{1}{t}) + \frac{1}{t}\cdot\|y-g\|$, and so $f(t) = \|y-g\| \ge t(2-\varepsilon) - t + 1 = 1 + t - \varepsilon t$; therefore,

$$\frac{\|y+x\|}{\|x\|+1} \ge \frac{1+t-\varepsilon t-\varepsilon}{1+t+\varepsilon}.$$

The last function increases with t, so we obtain

$$\frac{\|y+x\|}{\|x\|+1} \ge 1 - \frac{3\varepsilon}{2+\varepsilon} > 1 - \frac{3\varepsilon}{1+\varepsilon}.$$

Therefore, if $\frac{3\varepsilon}{1+\varepsilon} < \eta \left(\Leftrightarrow \varepsilon < \frac{\eta}{3-\eta}\right)$ we obtain: $\frac{\|y+x\|}{\|x\|+1} \le 1-\eta$, which concludes the proof. □

REMARK 1. Our result proves that X contains (isomorphically) ℓ_1 if and only if there is a renorming of X for which $\mu_2(X) = 2$. This shows that the condition $\mu_2(X) < 2$ is not invariant for renormings: for example, $\mu_2(\ell_\infty) = 3/2$ (see [8], Proposition 5); but since it contains ℓ_1, the space ℓ_∞ has a renorming which is octahedral ($\mu_2 = 2$).

But something different can be said.

Recall the following proposition (see [1], Theorem 8.1).

PROPOSITION 2.2. *For any space X, we have*

$$[\mu_1(X^{**}), \mu_2(X^{**})] \subset [\mu_1(X), \mu_2(X)] \tag{2.2}$$

We know that $\mu_2(c_0) = 3/2$ (see [8]), and of course $\mu_2(X) < 2$ for any X obtained by renorming c_0 (since c_0 does not contain ℓ_1); same results for the space c. For ℓ_∞ (the bidual of c_0 and c), according to Proposition 2.3, we must have $\mu_2(\ell_\infty) \le 3/2$ (in fact, equality holds). Also: if ℓ_{00} is the bidual of c_0 renormed in some way, then we must have $\mu_2(\ell_{00}) < 2$. The renorming of ℓ_∞ for which $\mu_2 = 2$ is not "a bidual norm": in fact (according to (3.2)), the predual should contain ℓ_1. But $|\ell_1^{**}| = 2^c$ and $|\ell_\infty| = c$.

The following *question* was raised in [5], p. 12: if X contains ℓ_1, does there exist a renorming of X such that the bidual norm is octahedral in X^{**}?
If the answer to the above question is yes, then according to (2.2) the renorming of X must be octahedral.

3. Octahedral norms and vicinities

Consider now the following conditions:
d) $\mu_2(X) = 2$;
e) X contains ℓ_1 isomorphically;
h) X contains ℓ_1 isometrically.
k) $\mu_2(X) = 2$ and such value is attained.
The following implications (and no others) hold:

$$\begin{array}{ccccc} & \Rightarrow & (h) & \Rightarrow & \\ (k) & & & & (e) \\ & \Rightarrow & (d) & \Rightarrow & \end{array}$$

The space ℓ_1 shows that d) and h) together do not imply k) (see [8]).
k) $\Rightarrow$ h) was proved in [9], Proposition 2.
e) does not imply d): see [2], Section 4.
The following examples A and B will show that conditions d) and h) are independent.

EXAMPLE 1. Let $X = \ell_1$ endowed with the following (strictly convex) norm: if $x = (x_1, \cdots, x_n, \cdots)$, set $|||x||| = \sum_{n=1}^{\infty} |x_n| + \left(\sum_{n=1}^{\infty} \frac{|x_n|^2}{2^n}\right)^{1/2}$.
This space does not contain ℓ_1 isometrically, and nevertheless $\mu_2(X) = 2$; this shows that d) does not imply h).

EXAMPLE 2. Take $K = \{c\} \cup [a, b]$ with $c \notin [a, b]$; then for $X = C(K)$, which contains ℓ_1 isometrically, we have $\mu_1(X) = \mu_2(X) = 3/2 < 2$ (see [9], Proposition 3). This shows that h) does not imply d).

4. Average distances and product of spaces with the max norm

In this section we indicate some results concerning μ_1 and μ_2 when the product of two spaces is done in the sense of ℓ_∞.

THEOREM 4.1. *Let $Z = (X \oplus Y)_\infty$. Then:*

$$\mu_2(Z) \geq \min(\mu_2(X), \mu_2(Y)). \tag{4.1}$$

Proof. Consider any $F = \{z_1, z_2, \cdots, z_n\} \subset S_Z$, with $z_i = (x_i, y_i)$ $(i = 1, 2, \cdots n)$. Then for every i we have either $x_i \in S_X$ or $y_i \in S_Y$: assume that (if any) $x_1, \cdots, x_k \in S_X$ and $y_{k+1}, \cdots, y_n \in S_Y$ $(0 \leq k \leq n)$; if $0 < k < n$, set $F_1 = \{x_1, \cdots, x_k\}$; $F_2 = \{y_{k+1}, \cdots, y_n\}$.
Let $x \in S_X$, $y \in S_Y$, so $z = (x, y) \in S_Z$; set $\mu(F_1, x) = \alpha$; $\mu(F_2, y) = \beta$.
We obtain:

$$\begin{aligned}
&n\mu_2(F) \geq n\mu(F, z)\\
&= \sum_{i=1}^{n} \|(x, y) - (x_i, y_i)\| = \sum_{i=1}^{n} \max(\|x - x_i\|, \|y - y_i\|)\\
&\geq \sum_{i=1}^{k} \|x - x_i\| + \sum_{i=k+1}^{n} \|y - y_i\| = k\alpha + (n-k)\beta\\
&\geq n \cdot \min\{\alpha, \beta\}.
\end{aligned}$$

Since $\mu_2(X) \leq \mu_2(F_1)$, we can choose x so that $\alpha \cong \mu_2(X)$; similarly, we can choose y so that $\beta \cong \mu_2(Y)$: we thus obtain $\mu_2(F_2) \geq \min(\alpha, \beta) \cong \min\{\mu_2(X), \mu_2(Y)\}$.
If $k = 0$ or $k = n$, then $F = F_2$ of $F = F_1$: so in any case $\mu_2(F) \geq \mu_2(Y)$ or $\mu_2(F) \geq \mu_2(X)$.
In any case, since one of the two inequalities is true for $F \subset S_Z$ arbitrary, we obtain $\mu_2(Z) \geq \min\{\mu_2(X), \mu_2(Y)\}$ which is (4.1). □

REMARK 2. The estimate given by (4.1) is sharp (we have equality in many simple cases).
In particular, let X or Y contain ℓ_1, so according to Theorem 2.2 we can find an octahedral norm in one of these spaces; then the product $(X \oplus Y)_\infty$ gives automatically an octahedral norm for the product space.

THEOREM 4.2. *Let* $Z = (X \oplus Y)_\infty$. *Then:*

$$\mu_1(Z) \geq \min(\mu_1(X), \mu_1(Y)). \tag{4.2}$$

Proof. For any $\varepsilon > o$, we can find sets $F_1 \subset S_X$, $F_2 \subset S_Y$, such that

$$\mu_1(F_1) \geq \mu_1(X) - \varepsilon; \quad \mu_1(F_2) \geq \mu_1(Y) - \varepsilon.$$

Without loss of generality, we can assume that F_1 and F_2 contain the same number of elements: in fact, let F_1 contain k elements and F_2 contain h elements; then we can "count" h times each element of F_1 and k times each element of F_2, so as to obtain "sets" F_1' and F_2' having each $n = kh$ elements (and the same value as F_1, F_2 for μ_1). Let $z_i = (x_i, y_i)$ with $x_i \in F_1'$, $y_i \in F_2'$, $i = 1, 2, \cdots, n$; then set $F = \{z_1, \cdots, z_n\}$: of course, $F \subset S_Z$.
Take now any element $z = (x, y) \in S_Z$; we must have either $x \in S_X$ or $y \in S_Y$. In the first case we obtain:

$$\begin{aligned}\mu(F, z) &= \frac{1}{n}\sum_{i=1}^{n} \max(\|x - x_i\|, \|y - y_i\|) \\ &\geq \frac{1}{n}(\|x - x_1\| + \cdots + \|x - x_n\|) = \mu_1(F_1', x) \\ &\geq \mu_1(F_1') \geq \mu_1(X) - \varepsilon;\end{aligned}$$

similarly, if $\|y\| = 1$, we obtain $\mu(F, z) \geq \mu_1(Y) - \varepsilon$. In any case, we obtain (for any element $z \in S_Z$) :

$$\mu(F, z) \geq \min(\mu_1(X), \mu_1(Y)) - \varepsilon,$$

which implies $\mu_1(Z) \geq \mu_1(F) \geq \min(\mu_1(X), \mu_1(Y)) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves (4.2). □

REMARK 3. According to Theorem 4.2, $\mu_1(Z) = 1$ ($Z = (X \oplus Y)_\infty$), implies $\mu_1(X) = 1$ or $\mu_1(Y) = 1$.

5. Average distances and product of spaces with the "sum" norm

In this section we indicate some estimates concerning μ_1 and μ_2 when the product is done in the sense of ℓ_1.

THEOREM 5.1. *Let* $Z = (X \oplus Y)_1$, *with* $\min(\mu_2(X), \mu_2(Y)) < 2$. *Then:*

$$\mu_2(Z) \leq \frac{4 - \mu_2(X)\mu_2(Y)}{4 - \mu_2(X) - \mu_2(Y)}. \tag{5.1}$$

Proof. Given any $\varepsilon > 0$, we can find finite sets $F_1 = \{x_1, x_2, \cdots, x_n\} \subset S_X$ and $F_2 = \{y_1, y_2, \cdots, y_n\} \subset S_Y$ such that

$$\mu_2(F_1) < \mu_2(X) + \varepsilon; \quad \mu_2(F_2) < \mu_2(Y) + \varepsilon. \tag{5.2}$$

It is not a restriction to assume that F_1 and F_2 have the same number of elements (for example, if they have respectively h and k elements, we may "count" k times each element of F_1 and h times each element of F_2 to obtain new "sets" with $n = h \cdot k$ elements each).
Now take $\alpha \in (0,1)$, then set $F = \{z_1, z_2, \cdots, z_n\}$, where, for each $i = 1, 2, \cdots, n : z_i = (\alpha x_i, (1-\alpha) y_i)$.
Consider now in S_Z a point $z = (x, y) : x \in X; \quad y \in Y; \|x\| + \|y\| = 1$. Set $\|x\| = \lambda$ and $\|y\| = 1 - \lambda$. We have:

$$\begin{aligned} n\mu(F, z) &= \sum_{i=1}^{n} \|(\alpha x_i, (1-\alpha) y_i) - (x, y)\| \\ &= \sum_{i=1}^{n} \|\alpha x_i - x\| + \sum_{i=1}^{n} \|(1-\alpha) y_i - y\| \\ &= \alpha \sum_{i=1}^{n} \|x_i - \frac{x}{\alpha}\| + (1-\alpha) \sum_{i=1}^{n} \|y_i - \frac{y}{1-\alpha}\|. \end{aligned} \tag{5.3}$$

Suppose now that

$$\|x\| = \lambda \geq \alpha, \text{ so } \|y\| = 1 - \lambda \leq 1 - \alpha; \tag{i}$$

we obtain:

$$\begin{aligned} \sum_{i=1}^{n} \left\|x_i - \frac{x}{\alpha}\right\| &\leq \sum_{i=1}^{n} \left(\left\|x_i - \frac{x}{\|x\|}\right\| + \left\|\frac{x}{\|x\|} - \frac{x}{\alpha}\right\| \right) \\ &\leq n \left(\mu_2(F_1) + \frac{\|x\|}{\alpha} - 1 \right); \end{aligned}$$

now observe that the function (of $t \in R$) $f(t) = \sum_{i=1}^{n} \|y_i - t \frac{y}{\|y\|}\|$ is convex; moreover, $f(0) = n$ and $f(1) \leq n\mu_2(F_2)$. Since $\frac{y}{1-\alpha}$ can be expressed as a convex combination of Θ and $\frac{y}{\|y\|}$ (namely, $\frac{y}{1-\alpha} = \left(1 - \frac{1-\lambda}{1-\alpha}\right)\Theta + \frac{1-\lambda}{1-\alpha}\frac{y}{\|y\|}$) we obtain:

$$\sum_{i=1}^{n} \|y_i - \frac{y}{1-\alpha}\| \leq n\left(1 - \frac{1-\lambda}{1-\alpha} + \frac{1-\lambda}{1-\alpha}\mu_2(F_2)\right).$$

Thus $n\mu(F,z) = \alpha\sum_{i=1}^{n}\|x_i - \frac{x}{\alpha}\| + (1-\alpha)\sum_{i=1}^{n}\|y_i - \frac{y}{1-\alpha}\| \leq n\alpha(\mu_2(F_1) + \frac{\lambda}{\alpha} - 1) + n(1-\alpha)(1 - \frac{1-\lambda}{1-\alpha} + \frac{1-\lambda}{1-\alpha}\mu_2(F_2)) = n[\alpha\mu_2(F_1) + \lambda - \alpha + \lambda - \alpha + (1-\lambda)\mu_2(F_2)]$; this implies:

$$(i') \qquad \begin{aligned} \mu(F,z) &\leq \mu_2(F_2) + \alpha(\mu_2(F_1) - 2) + \lambda(2 - \mu_2(F_2)) \\ &\leq \alpha\mu_2(F_1) - 2\alpha + 2 \quad (\text{since } \lambda \leq 1). \end{aligned}$$

Now suppose instead that

$$(ii) \qquad \|x\| = \lambda < \alpha, \quad \text{so } \|y\| = 1 - \lambda > 1 - \alpha;$$

with a similar reasoning, we obtain:

$$\sum_{i=1}^{n} \left\|y_i - \frac{y}{1-\alpha}\right\| \leq n\left(\mu_2(F_2) + \frac{\|y\|}{1-\alpha} - 1\right);$$

$$\sum_{i=1}^{n} \left\|x_i - \frac{x}{\alpha}\right\| \leq n\left(1 - \frac{\lambda}{\alpha} + \frac{\lambda}{\alpha}\mu_2(F_1)\right),$$

and then

$$\begin{aligned} n\mu(F,z) &= \alpha\sum_{i=1}^{n}\left\|x_i - \frac{x}{\alpha}\right\| + (1-\alpha)\sum_{i=1}^{n}\left\|y_i - \frac{y}{1-\alpha}\right\| \\ &\leq n[\alpha - \lambda + \lambda\mu_2(F_1) + (1-\alpha)\mu_2(F_2) + 1 - \lambda - 1 + \alpha]; \end{aligned}$$

this implies

$$(ii') \qquad \begin{aligned} \mu(F,z) &\leq \mu_2(F_2) + \alpha(2 - \mu_2(F_2)) + \lambda(\mu_2(F_1) - 2) \\ &\leq \mu_2(F_2) + \alpha(2 - \mu_2(F_2)). \end{aligned}$$

Therefore, for any $z \in S(Z)$ (see (i'), (ii')):

$$\mu(F,z) \leq \sup(\alpha\mu_2(F_1) - 2\alpha + 2,\ \mu_2(F_2)(1-\alpha) + 2\alpha),$$

so

$$\mu_2(F) \leq \sup(2 + \alpha\mu_2(F_1) - 2\alpha,\ \mu_2(F_2)(1-\alpha) + 2\alpha).$$

If we choose α so that $\alpha(4-\mu_2(F_2)-\mu_2(F_1)) = 2-\mu_2(F_2)$, thus $1-\alpha = \frac{2-\mu_2(F_1)}{4-\mu_2(F_2)-\mu_2(F_1)}$, then we obtain

$$\mu_2(Z) \leq \mu_2(F) \leq \frac{4-\mu_2(F_1)\mu_2(F_2)}{4-\mu_2(F_1)-\mu_2(F_2)}.$$

According to (5.2), since the ε chosen at the beginning can be arbitrarily small, this implies the thesis. □

REMARK 4. Inequality (5.1) is meaningful, in the sense that it gives $\mu_2((X \oplus Y)_1) < 2$, whenever $\max(\mu_2(X), \mu_2(Y)) < 2$.

THEOREM 5.2. *Let* $Z = (X \oplus Y)_1$, *with* $1 < \min(\mu_1(X), \mu_1(Y)) < 2$. *Then:*

$$\mu_1(Z) \geq \frac{4\mu_1(X)\mu_1(Y) - \mu_1^2(X)\mu_1^2(Y)}{4[\mu_1(X)+\mu_1(Y)-\mu_1(X)\mu_1(Y)]}. \tag{5.4}$$

Proof. For any $\varepsilon > 0$, we can find sets $F_1 = \{x_1, x_2, \cdots, x_n\} \subset S_X$ and $F_2 = \{y_1, y_2, \cdots, y_n\} \subset S_Y$ such that

$$\mu_1(F_1) > \mu_1(X) - \varepsilon; \quad \mu_1(F_2) > \mu_1(Y) - \varepsilon; \tag{5.5}$$

once again, we observe that it is not a restriction to assume that F_1 and F_2 have the same number of elements. Now take $\alpha \in (0,1)$, then set $F = \{z_1, z_2, \cdots, z_n\}$, where, for each $i = 1, 2, \cdots, n:\ z_i = (\alpha x_i, (1-\alpha)y_i)$. Consider now in S_Z a point $z = (x, y):\ x \in X; y \in Y; \|x\| + \|y\| = 1$. Set $\|x\| = \lambda$ and $\|y\| = 1-\lambda$. Again, we can use (5.3); also, to simplify the notations set

$$\frac{1}{n}\sum_{i=1}^{n} \|x_i - \frac{x}{\|x\|}\| = \mu_x; \quad \frac{1}{n}\sum_{i=1}^{n} \|y_i - \frac{y}{\|y\|}\| = \mu_y. \tag{j}$$

Suppose now that

$$\|x\| = \lambda \geq \alpha, \text{ so } \|y\| = 1-\lambda \leq 1-\alpha; \tag{i}$$

note that the slope of the convex function $f(t) = \sum_{i=1}^{n} \|x_i - t\frac{x}{\|x\|}\|$, for $t \geq 1$, is at least $n(\mu_x - 1)$, so

$$\sum_{i=1}^{n} \left\|x_i - \frac{x}{\alpha}\right\| \geq n\left(\mu_x + (\mu_x - 1)\left(\frac{\|x\|}{\alpha} - 1\right)\right); \tag{5.6}$$

concerning $\sum_{i=1}^{n}\|y_i - \frac{y}{1-\alpha}\|$ two estimates are possible (since the slope of the convex function $f(t) = \sum_{i=1}^{n}\|y_i - t\frac{y}{\|y\|}\|$, for $0 < t < 1$, is not larger than n, while $f(0) = n$); we have:

$$\sum_{i=1}^{n}\|y_i - \frac{y}{1-\alpha}\| \geq n \max\left(\mu_y - \left(1 - \frac{\|y\|}{1-\alpha}\right), \frac{\mu_y}{2}\right).$$

Therefore, under assumption (i), we obtain (according to (5.3)):

$$\begin{aligned}\mu(F,z) &\geq \alpha\left(\mu_x + (\mu_x - 1)\left(\frac{\lambda}{\alpha} - 1\right)\right)\\ &\quad +(1-\alpha)\max\left(\mu_y - \left(1 - \frac{1-\lambda}{1-\alpha}\right), \frac{\mu_y}{2}\right)\\ &= \lambda(\mu_x - 1) + \alpha + \max\left((1-\alpha)\mu_y + \alpha - \lambda, (1-\alpha)\frac{\mu_y}{2}\right).\end{aligned}$$

Note that

$$(f)\quad \max\left((1-\alpha)\mu_y + \alpha - \lambda, (1-\alpha)\tfrac{\mu_y}{2}\right)$$

$$= \begin{cases} (1-\alpha)\mu_y + \alpha - \lambda & \text{if } \alpha \leq \lambda \leq \alpha + (1-\alpha)\frac{\mu_y}{2} \qquad (f_1)\\ (1-\alpha)\frac{\mu_y}{2} & \text{if } \alpha + (1-\alpha)\frac{\mu_y}{2} \leq \lambda \leq 1 \qquad (f_2).\end{cases}$$

In case (f_1) holds, we obtain:

$$\begin{aligned}\mu(F,z) &\geq \lambda(\mu_x - 1) + \alpha + (1-\alpha)\mu_y + \alpha - \lambda\\ &\geq (1-\alpha)\mu_y + 2\alpha + \left(\alpha + (1-\alpha)\frac{\mu_y}{2}\right)(\mu_x - 2)\\ &= \alpha\mu_x + \frac{1-\alpha}{2}\mu_x\mu_y.\end{aligned}$$

Also in case (f_2) holds, we obtain;

$$\begin{aligned}\mu(F,z) &\geq \lambda(\mu_x - 1) + \alpha + (1-\alpha)\frac{\mu_y}{2}\\ &\geq \left(\alpha + (1-\alpha)\frac{\mu_y}{2}\right)\cdot(\mu_x - 1) + \alpha + (1-\alpha)\cdot\frac{\mu_y}{2}\\ &= \alpha\mu_x + \frac{1-\alpha}{2}\mu_x\mu_y.\end{aligned}$$

Suppose instead now that

$$(ii)\qquad \|x\| = \lambda < \alpha, \quad \text{so } \|y\| = 1 - \lambda > 1 - \alpha;$$

with similar reasoning, we obtain:

$$\sum_{i=1}^{n} \|y_i - \frac{y}{1-\alpha}\| \geq n\left(\mu_y + (\mu_y - 1)\left(\frac{\|y\|}{1-\alpha} - 1\right)\right);$$

Also, concerning $\sum_{i=1}^{n} \|x_i - \frac{x}{\alpha}\|$, two estimates are possible, and we have:

$$\sum_{i=1}^{n} \|x_i - \frac{x}{\alpha}\| \geq n \max\left(\mu_x - \left(1 - \frac{\|x\|}{\alpha}\right), \frac{\mu_x}{2}\right).$$

Therefore, under assumption (ii), we obtain (according to (5.3) and with the notations (j)):

$$\begin{aligned}
&\mu(F,z)\\
&\geq \alpha\left(\max\left(\mu_x - \left(1 - \frac{\lambda}{\alpha}\right), \frac{\mu_x}{2}\right)\right.\\
&\qquad \left. +(1-\alpha)(\mu_y + (\mu_y - 1)\left(\frac{1-\lambda}{1-\alpha} - 1\right)\right)\\
&= \max\left(\alpha\mu_x - \alpha + \lambda, \frac{\alpha\mu_x}{2}\right) + (1-\lambda)\mu_y + \lambda - \alpha.
\end{aligned}$$

Note that:

(g) $\quad \max\left(\alpha\mu_x - \alpha + \lambda, \frac{\alpha\mu_x}{2}\right)$

$$= \begin{cases} \alpha\mu_x - \alpha + \lambda, & \text{if } \alpha(1 - \frac{\mu_x}{2}) \leq \lambda < \alpha \qquad (g_1) \\ \frac{\alpha\mu_x}{2}, & \text{if } 0 \leq \lambda \leq \alpha(1 - \frac{\mu_x}{2}). \qquad (g_2) \end{cases}$$

In case (g_1) holds, we obtain:

$$\begin{aligned}
\mu(F,z) &\geq \alpha\mu_x - \alpha + \lambda + (1-\lambda)\mu_y + \lambda - \alpha\\
&\geq \alpha\mu_x + \mu_y - 2\alpha + \alpha\left(1 - \frac{\mu_x}{2}\right)(2 - \mu_y)\\
&= \mu_y(1-\alpha) + \mu_x\mu_y\frac{\alpha}{2}.
\end{aligned}$$

Also in case (g_2) holds, we obtain:

$$\begin{aligned}\mu(F,z) &\geq \frac{\alpha\mu_x}{2} + (1-\lambda)\mu_y + \lambda - \alpha \\ &\geq \frac{\alpha\mu_x}{2} + \mu_y - \alpha + \alpha\left(1 - \frac{\mu_x}{2}\right)(1-\mu_y) \\ &= \mu_y(1-\alpha) + \mu_x\mu_y\frac{\alpha}{2}.\end{aligned}$$

Therefore, we can say (since z is arbitrary) that we have

$$\mu(F,z) \geq \inf\left(\alpha\mu_x + \frac{1-\alpha}{2}\mu_x\mu_y,\ \mu_y(1-\alpha) + \mu_x\mu_y\frac{\alpha}{2}\right).$$

But $\mu_x \geq \mu_1(F_1)$ and $\mu_y \geq \mu_1(F_2)$, thus, for every z we have:

$$\begin{aligned}&\mu(F,z) \\ &\geq \inf\left(\alpha\mu_1(F_1) + \frac{1-\alpha}{2}\mu_1(F_1)\mu_1(F_2),\ (1-\alpha)\mu_1(F_2) + \mu_1(F_1)\mu_1(F_2)\frac{\alpha}{2}\right),\end{aligned}$$

so the same estimate is true for $\mu_1(F)$.
Now we can choose α so that

$$\alpha\mu_1(F_1) + \frac{1-\alpha}{2}\mu_1(F_1)\mu_1(F_2) = (1-\alpha)\mu_1(F_2) + \mu_1(F_1)\mu_1(F_2)\frac{\alpha}{2}$$

(if $\mu_1 = \mu_1(F_1)$, $\mu_2 = \mu_1(F_2)$, then $\alpha = [2\mu_2 - \mu_1\mu_2] : [2(\mu_1 + \mu_2 - \mu_1\mu_2)]$; so $1 - \alpha = [2\mu_1 - \mu_1\mu_2] : [2(\mu_1 + \mu_2 - \mu_1\mu_2)]$).
Also, recall that F_1 and F_2 can be chosen so that $\mu_1(F_1)$ is very near to $\mu_1(X)$ and $\mu_1(F_2)$ is very near to $\mu_1(Y)$ (see (5.5)); so, finally, we obtain:

$$\mu_1(Z) \geq \mu_1(F) \geq \frac{4\mu_1(X)\mu_1(Y) - \mu_1^2(X)\mu_1^2(Y)}{4[\mu_1(X) + \mu_1(Y) - \mu_1(X)\mu_1(Y)]}, \text{which is (5.4).} \qquad \square$$

REMARK 5. For example, if $\mu_1(X) = \mu_1(Y) = k$, the above estimate is meaningful (it gives $\mu_1(Z) > 1$) for $k^3 - 8k + 8 < 0$, so at least when $k > \sqrt{5} - 1$.

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