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# LOCAL DERIVATIONS OF THE POLYNOMIAL RING OVER A FIELD

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ABSTRACT. In this article, we give an example of local derivation, that is not derivation, on the algebra $F(x_1, \cdots, x_n)$ of rational functions in $x_1, \cdots, x_n$ over an infinite field $F$, and show that if $X$ is a set of symbols and $\{x_1, \cdots, x_n\}$ is a finite subset of $X$, $n \geq 1$, then each local derivation of $F[x_1, \cdots, x_n]$ into $F[X]$ is a $F$-derivation and each local derivation of $F[X]$ into itself is also a $F$-derivation.

## 1. Introduction

Let $R$ be a commutative ring with unity 1 and $X$ a set of symbols. The *free monoid* on $X$ is the set $G_X$ of all finite sequences $x_1 \cdots x_n$ of elements from $X$, including the empty sequence, with the multiplication defined by juxtaposing sequences: $(x_1 \cdots x_n)(y_1 \cdots y_m) = x_1 \cdots x_n y_1 \cdots y_m$, $x_i, y_j \in X$ $(i = 1, \cdots, n,\ j = 1, \cdots, m)$. Thus, the empty sequence is the unity element of $G_X$. The *free R-algebra* on $X$ is the $R$-algebra $RG_X$ satisfying that every mapping from $X$ to an $R$-algebra $A$ extends uniquely to an $R$-algebra homomorphism of $RG_X$ to $A$. The familiar *commutative polynomial ring* $R[X]$ can be obtained by a similar construction: let $H_X$ be the free commutative monoid on $X$. Then $R[X] = RH_X$. If $X$ consists of the distinct symbols $x_1, \cdots, x_n$, $n \geq 1$, then we will write $R[x_1, \cdots, x_n]$ for $R[X]$ (see [4, p. 6]).

Let $A$ be a commutative $R$-algebra with unity 1 and $M$ an $A$-module. An $R$-linear mapping $d : A \to M$ is called an *R-derivation of A into M* if $d(ab) = ad(b) + bd(a)$, $a, b \in A$, and an $R$-derivation $d : A \to A$ is called an *R-derivation on A*. A couple $(M, d)$ is called a *derivation module* of $A$ if $M$ is an $A$-module and $d$ is an $R$-derivation of $A$ into $M$. Let

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$(M, d)$ and $(N, \delta)$ be derivation modules of $A$. Then an $A$-module homomorphism $f : M \to N$ is called a *derivation module homomorphism* if $f \circ d = \delta$, and a derivation module homomorphism which is one-to-one and onto is called a *derivation module isomorphism.* A derivation module $(U, d)$ of $A$ is said to be *universal* if for any derivation module $(M, \delta)$ of $A$, there exists a unique derivation module homomorphism $f : (U, d) \to (M, \delta)$.

It is well known (see [2]) that for any commutative $R$-algebra $A$ with unity 1, there exists a universal derivation module of $A$, and it is unique up to derivation module isomorphisms. And a universal derivation module $(U, d)$ of $A$ can be constructed in the following way: let $U = A \otimes_R A/J$, where $J$ is the $A$-submodule of $A \otimes_R A$ generated by all elements of the form $1 \otimes ab - a \otimes b - b \otimes a$, $a, b \in A$, and define $d : A \to U$ by $d(a) = \nu(1 \otimes a)$, $a \in A$, where $\nu : A \otimes_R A \to U$ is the natural module homomorphism. Then $(U, d)$ is a universal derivation module of $A$.

An $R$-linear mapping $\alpha : A \to M$ is called a *local derivation of $A$ into $M$* if for each $a \in A$, there exists an $R$-derivation $\delta_a : A \to M$ such that $\alpha(a) = \delta_a(a)$, and a local derivation $\alpha : A \to A$ is called a *local derivation on $A$*.

Every derivation is a local derivation but the converse is not true, in general. The example is constructed by C. U. Jensen (in response to question raised during a recture in Copenhagen in 1986). This example uses the algebra $\mathbb{C}(x)$ of rational functions in one variable $x$ over the field $\mathbb{C}$ of complex numbers [3]. This adds further interest to determining the local derivations of the polynomial rings over $\mathbb{C}$ and any field $F$. Richard V. Kadison showed that each local derivation of $\mathbb{C}[x_1, \cdots, x_n]$ into $\mathbb{C}[x_1, \cdots, x_m]$ is a $\mathbb{C}$-derivation, where $n$ and $m$ are positive integers with $1 \leq n \leq m$ [3]. And the similar results for the polynomial rings over an algebraically closed field are proved in [5, p. 32-41].

In this paper, with any infinite field $F$, we show that a $F$-linear mapping $\alpha$ of $F(x_1, \cdots, x_n)$ into itself is a local derivation if and only if $\alpha(c) = 0$ for all constant $c \in F(x_1, \cdots, x_n)$, where $F(x_1, \cdots, x_n)$ is the algebra of rational functions in $x_1, \cdots, x_n$ over $F$, and give an example of a local derivation on $F(x_1, \cdots, x_n)$ that is not $F$-derivation. Next, it is proved that if $\{x_1, \cdots, x_n\}$ $(n \geq 1)$ is a finite subset of a set

$X$ of symbols, then each local derivation of $F[x_1, \cdots, x_n]$, and of $F[X]$, into $F[X]$ is a $F$-derivation. Throughout this paper, $F$ is an infinite field and $X$ is a set of symbols.

## 2. Local derivations of the algebra of rational functions over a field

In this section, $F(x_1, \cdots, x_n)$ will denote the algebra of rational functions over $F$ in a set $\{x_1, \cdots, x_n\}$ of variables.

PROPOSITION 2.1 ([1]). *Let $R[X]$ be a polynomial ring over a commutative ring $R$ with unity. If $U$ is a free $R[X]$-module with a basis $\{u_x : x \in X\}$, where $u_x = u_y$ if and only if $x = y$ for $x, y \in X$, and $d : R[X] \to U$ is an $R$-derivation defined by $d(f) = \sum_{x \in X} (\partial f / \partial x) u_x$, $f \in R[X]$, then $(U, d)$ is a universal derivation module of $R[X]$.*

Let $(U, d)$ be the universal derivation module (in Proposition 2.1) of $F[x_1, \cdots, x_n]$. Then $d(x_i) = \sum\limits_{j=1}^{n} \frac{\partial x_i}{\partial x_j} u_{x_j} = \frac{\partial x_i}{\partial x_i} u_{x_i} = u_{x_i}$, $i \in \{1, \cdots, n\}$. If $\delta$ is a $F$-derivation on $F[x_1, \cdots x_n]$, then there exists a unique $F[x_1, \cdots, x_n]$-module homomorphism $\phi : U \to F[x_1, \cdots, x_n]$ such that $\phi \circ d = \delta$, whence $\delta(x_i) = \phi(d(x_i)) = \phi(u_{x_i})$, $i \in \{1, \cdots, n\}$, and we have that

$$\delta(f) = \phi(d(f)) = \phi(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} u_{x_i}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \phi(u_{x_i}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta(x_i),$$

$f \in F[x_1, \cdots, x_n]$. Hence for any $F$-derivation $\delta$ on $F[x_1, \cdots, x_n]$, $\delta(f) = \sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta(x_i)$, $f \in F[x_1, \cdots, x_n]$.

LEMMA 2.2. *Let $\delta$ be a mapping of $F(x_1, \cdots, x_n)$ into itself. Then the following two conditions are equivalent:*

(1) $\delta(f) = \sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta(x_i)$ *for all* $f \in F(x_1, \cdots, x_n)$,

(2) $\delta$ *is a $F$-derivation.*

*Proof.* Let $\delta : F(x_1, \cdots, x_n) \to F(x_1, \cdots, x_n)$ be a mapping satisfying that $\delta(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\delta(x_i)$, $f \in F(x_1, \cdots, x_n)$. Then $\delta$ is a $F$-linear mapping and we have that

$$\begin{aligned}\delta(fg) &= \sum_{i=1}^{n} \frac{\partial(fg)}{\partial x_i}\delta(x_i) = \sum_{i=1}^{n}\left(f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}\right)\delta(x_i)\\ &= f\left(\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}\delta(x_i)\right) + g\left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\delta(x_i)\right)\\ &= f\delta(g) + g\delta(f),\end{aligned}$$

$f, g \in F(x_1, \cdots, x_n)$. Hence $\delta$ is a $F$-derivation on $F(x_1, \cdots, x_n)$.

Conversely, let $\delta$ be a $F$-derivation on $F(x_1, \cdots, x_n)$, and if $p$ is a non-zero polynomial in $F[x_1, \cdots, x_n]$, then

$$0 = \delta(1) = \delta(pp^{-1}) = \delta(p)p^{-1} + p\delta(p^{-1}),$$

whence $\delta(p^{-1}) = -\delta(p)p^{-2}$. Thus, for any rational function $f = pq^{-1}$ in $F(x_1, \cdots, x_n)$, where $p, q \in F[x_1, \cdots, x_n]$ and $q \neq 0$, we have that

$$\begin{aligned}\delta(f) &= \delta(pq^{-1}) = \delta(p)q^{-1} + p\delta(q^{-1}) = \delta(p)q^{-1} - p\delta(q)q^{-2}\\ &= \left[\sum_{i=1}^{n}\frac{\partial p}{\partial x_i}\delta(x_i)\right]qq^{-2} - p\left[\sum_{i=1}^{n}\frac{\partial q}{\partial x_i}\delta(x_i)\right]q^{-2}\\ &= \sum_{i=1}^{n}\left[\left(q\frac{\partial p}{\partial x_i} - p\frac{\partial q}{\partial x_i}\right)q^{-2}\right]\delta(x_i)\\ &= \sum_{i=1}^{n}\frac{\partial(pq^{-1})}{\partial x_i}\delta(x_i) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\delta(x_i).\end{aligned}$$

□

THEOREM 2.3. *Let $\alpha$ be a $F$-linear mapping of $F(x_1, \cdots, x_n)$ into itself. Then the following two conditions are equivalent:*

(1) *$\alpha$ is a local derivation,*

(2) *$\alpha(c) = 0$ for all constant $c \in F(x_1, \cdots, x_n)$.*

*Proof.* Let $\alpha$ be a local derivation on $F(x_1, \cdots, x_n)$. Then for each constant $c \in F(x_1, \cdots, x_n)$, there is a $F$-derivation $\delta_c$ on $F(x_1, \cdots, x_n)$ such that $\alpha(c) = \delta_c(c) = 0$.

Conversely, if $f$ is a constant in $F(x_1, \cdots, x_n)$, then for any $F$-derivation $\delta$ on $F(x_1, \cdots, x_n)$, $\alpha(f) = 0 = \delta(f)$. Let $f$ be a non-constant in $F(x_1, \cdots, x_n)$. Then there exists a $x_i$ in $\{x_1, \cdots, x_n\}$ such that $\partial f/\partial x_i \neq 0$. Take one of such $x_i$ and we denote it by $x$, and define a mapping $\delta_f$ of $F(x_1, \cdots, x_n)$ into itself by following:

$$\delta_f(h) = \frac{\partial h/\partial x}{\partial f/\partial x}\alpha(f)$$

for each $h \in F(x_1, \cdots, x_n)$, then $\delta_f$ is a $F$-derivation and we have that

$$\delta_f(f) = \frac{\partial f/\partial x}{\partial f/\partial x}\alpha(f) = \alpha(f),$$

hence $\alpha$ is a local derivation. □

We have an example of a local derivation that is not derivation as follows: we can consider $F(x_1, \cdots, x_n)$ as a vector space over $F$. Let $X$ be the $(n+1)$-dimensional subspace of $F(x_1, \cdots, x_n)$ generated by $\{1, x_1, \cdots, x_n\}$, $Y$ a complement of $X$, and $\alpha : F(x_1, \cdots, x_n) \to F(x_1, \cdots, x_n)$ the projection on $Y$ along $X$. Then $\alpha(c) = 0$ for all constant $c \in F(x_1, \cdots, x_n)$, whence $\alpha$ is a local derivation by Theorem 2.3, but it is not $F$-derivation, in fact, if $\alpha$ is a $F$-derivation, then from Lemma 2.2, $\alpha(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\alpha(x_i)$ for all $f \in F(x_1, \cdots, x_n)$, and $\alpha(x_i) = 0$, since $x_i \in X$, $i \in \{1, \cdots, n\}$. Hence $\alpha(f) = 0$ for all $f \in F(x_1, \cdots, x_n)$, in contradiction with $\alpha \neq 0$.

## 3. Local derivations of the commutative polynomial algebra

If $\{x_1, \cdots, x_n\}$ is a finite subset of $X$ $(n \geq 1)$, then $F[X]$ is viewed as a $F[x_1, \cdots, x_n]$-module. Let $\alpha$ be a local derivation of $F[x]$ into $F[X]$ with $x \in X$. Then $\alpha(c) = 0$ for any constant $c \in F[x]$, and for each $k \in \mathbb{N}$ ($\mathbb{N}$ is the set of positive integers), there exists a $F$-derivation $\delta_{x^k}$

of $F[x]$ into $F[X]$ such that $\alpha(x^k) = \delta_{x^k}(x^k)$. We will denote the $\delta_{x^k}$ by $\delta_k$. It is easy to show that a $F$-linear mapping $\delta : F[x] \to F[X]$ is a $F$-derivation if and only if $\delta(f) = f'\delta(x)$ for all $f \in F[x]$, where $f'$ is the usual derivative of $f$, by applying the multiplicative property of the derivation.

LEMMA 3.1. *Let $\alpha$ be a local derivation of $F[x]$ into $F[X]$, where $x \in X$. If $\alpha(x) = \delta_k(x)$ for each $k \in \mathbb{N}$, where $\delta_k : F[x] \to F[X]$ is the $F$-derivation such that $\delta_k(x^k) = \alpha(x^k)$, $k \in \mathbb{N}$, then $\alpha$ is a $F$-derivation.*

*Proof.* Since $\alpha(x^k) = \delta_k(x^k) = kx^{k-1}\delta_k(x) = kx^{k-1}\alpha(x)$ for each $k \in \mathbb{N}$, we have that

$$\alpha(f) = (a_1 + 2a_2x + \cdots + na_nx^{n-1})\alpha(x) = f'\alpha(f),$$

for $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in F[x]$, where $f'$ is the usual derivative of $f$, hence $\alpha$ is a $F$-derivation. □

THEOREM 3.2. *Each local derivation of $F[x]$ into $F[X]$ is a $F$-derivation, where $x \in X$.*

*Proof.* Let $\alpha$ be a local derivation of $F[x]$ into $F[X]$ with $x \in X$. Then for each $j \in \mathbb{N}$, there is a $F$-derivation $\delta_j : F[x] \to F[X]$ such that

$$\alpha(x) = \delta_1(x) \text{ and } \alpha(x^j) = jx^{j-1}\delta_j(x) \ (j \geq 2).$$

Let $g_j = \delta_j(x)$, $j \in \mathbb{N}$. Then $\alpha(x^j) = jx^{j-1}g_j$, $j \in \mathbb{N}$, and we shall show that $g_1 = g_2 = \cdots = g_n = \cdots$, whence $\alpha$ is a $F$-derivation from Lemma 3.1.

Choose a non-zero element $a$ in $F$ and let $p = x^j - a^{j-k}x^k$, $k, j \in \mathbb{N}$. Then $p \in F[x]$ and there is a $F$-derivation $\delta_{p^2} : F[x] \to F[X]$ such that $\alpha(p^2) = \delta_{p^2}(p^2)$, hence we have that

$$\begin{aligned}(1)\qquad 2p\delta_{p^2}(p) = \delta_{p^2}(p^2) &= \alpha(p^2)\\ &= \alpha(x^{2j} - 2a^{j-k}x^{j+k} + a^{2(j-k)}x^{2k})\\ &= \alpha(x^{2j}) - 2a^{j-k}\alpha(x^{j+k}) + a^{2(j-k)}\alpha(x^{2k})\\ &= 2jx^{2j-1}g_{2j} - 2(j+k)a^{j-k}x^{j+k-1}g_{j+k}\\ &\quad + 2ka^{2(j-k)}x^{2k-1}g_{2k}.\end{aligned}$$

With $x$ replaced by $a$ in (1), we have that

$$0 = 2a^{2j-1}\left[jg_{2j} - (j+k)g_{j+k} + kg_{2k}\right]_{x=a},$$

since the left side of (1) is 0, for $p(a) = 0$, whence

$$0 = \left[jg_{2j} - (j+k)g_{j+k} + kg_{2k}\right]_{x=a}.$$

Since $a$ is arbitrary non-zero element in $F$,

$$0 = \left[jg_{2j} - (j+k)g_{j+k} + kg_{2k}\right]_{x=a}$$

for all non-zero $a \in F$. Hence $0 = jg_{2j} - (j+k)g_{j+k} + kg_{2k}$ in $F[X]$ and it follows that

(2) $$(j+k)g_{j+k} = jg_{2j} + kg_{2k}.$$

For any $k \in \mathbb{N}$ and non-zero $a \in F$, let $q = 1 - a^{-k}x^k$. Then $q \in F[x]$ and there is a $F$-derivation $\delta_{q^2} : F[x] \to F[X]$ such that $\alpha(q^2) = \delta_{q^2}(q^2)$. Proceeding as in the computation of (1), we have that

(3) $$2q\delta_{q^2}(q) = -2k(a^{-k}x^{k-1}g_k - a^{-2k}x^{2k-1}g_{2k}),$$

and with $x$ replaced by $a$ in (3), $0 = a^{-1}\left[g_k - g_{2k}\right]_{x=a}$, for $q(a) = 0$, hence $0 = [g_k - g_{2k}]_{x=a}$ for all non-zero $a \in F$, whence it follows that $g_k - g_{2k} = 0$ in $F[X]$ and

(4) $$g_k = g_{2k}.$$

If $j = k+1$ in (2), then we have that

(5) $$(2k+1)g_{2k+1} = (k+1)g_{2k+2} + kg_{2k}.$$

We shall show that $g_1 = g_2 = g_3 = \cdots = g_{2n-1} = g_{2n} = g_{2n+1} = \cdots$ by induction in $n$.

If $k = 1$ in (4) and (5), then $g_1 = g_2$ and $3g_3 = 2g_4 + g_2$, respectively. Since $g_2 = g_4$ by (4), $3g_3 = 3g_2$ and $g_3 = g_2$, hence $g_1 = g_2 = g_3$. Suppose that $g_1 = g_2 = g_3 = \cdots = g_{2k-1} = g_{2k} = g_{2k+1}$, $k \in \mathbb{N}$. Then

$g_{2k} = g_{2k+1}$, hence from (5), $(2k+1)g_{2k+1} = (k+1)g_{2k+2} + kg_{2k+1}$, whence $(k+1)g_{2k+1} = (k+1)g_{2k+2}$ and $g_{2k+1} = g_{2k+2}$, hence we have that $g_1 = g_2 = \cdots = g_{2k+1} = g_{2k+2}$. Since $g_{k+2} = g_{2k+2}$ and since $g_{k+2} = g_{2k+4}$ from (4), $g_{2k+2} = g_{2k+4}$. With $k$ replaced by $k+1$ in (5),

$$\begin{aligned}(2k+3)g_{2k+3} &= (k+2)g_{2k+4} + (k+1)g_{2k+2} \\ &= (k+2)g_{2k+2} + (k+1)g_{2k+2} \\ &= (2k+3)g_{2k+2}\end{aligned}$$

and $g_{2k+3} = g_{2k+2}$. Hence $g_1 = g_2 = \cdots = g_{2k+1} = g_{2k+2} = g_{2k+3}$. It follows by induction that $g_i = g_j$ for each $i, j \in \mathbb{N}$. □

LEMMA 3.3. *Let $\alpha$ be a local derivation of $F[x_1, \cdots, x_n]$ into $F[X]$ that satisfies $\alpha(x_j) = 0$ for each $j \in \{1, \cdots, n\}$, where $\{x_1, \cdots, x_n\}$ is a finite subset of $X$. If the restriction of $\alpha$ to $F[x_{j(1)}, \cdots, x_{j(n-1)}]$ is a $F$-derivation for all $(n-1)$-element subset $\{x_{j(1)}, \cdots, x_{j(n-1)}\}$ of $\{x_1, \cdots, x_n\}$, then the local derivation $\alpha$ is 0.*

*Proof.* Let $\alpha : F[x_1, \cdots, x_n] \to F[X]$ be a local derivation such that $\alpha(x_j) = 0$, $j \in \{1, \cdots, n\}$. Then since the restriction of $\alpha$ to $F[x_1]$ is a $F$-derivation from Theorem 3.2, we have that

$$\alpha(x_1^k) = kx_1^{k-1}\alpha(x_1) = 0, \tag{1}$$

$k \in \mathbb{N}$, and since the restriction of $\alpha$ to $F[x_2, \cdots, x_n]$ is $F$-derivation by hypothesis, we have that

$$\begin{aligned}&\alpha(x_2^{k(2)} \cdots x_n^{k(n)}) \\ &= \sum_{i=2}^{n} k(i) x_2^{k(2)} \cdots x_{i-1}^{k(i-1)} x_i^{k(i)-1} x_{i+1}^{k(i+1)} \cdots x_n^{k(n)} \alpha(x_i) \\ &= 0\end{aligned} \tag{2}$$

for all non-negative integers $k(i)$ $(i = 2, \cdots, n)$.

Let $x_1^{k(1)} \cdots x_n^{k(n)}$ be a monic monomial in $F[x_1, \cdots, x_n]$ with non-negative integers $k(i)$ $(i = 1, \cdots, n)$. Choose any non-zero elements

$b_1, \cdots, b_n$ in $F$, and let $a = -b_1^{k(1)} b_2^{-k(2)} \cdots b_n^{-k(n)}$ and $q = x_1^{k(1)} + a x_2^{k(2)} \cdots x_n^{k(n)}$. Then $a$ is a non-zero in $F$ and $q \in F[x_1, \cdots, x_n]$. Since $\alpha$ is a local derivation of $F[x_1, \cdots, x_n]$ into $F[X]$, there is a $F$-derivation $\delta_{q^2} : F[x_1, \cdots, x_n] \to F[X]$ such that $\alpha(q^2) = \delta_{q^2}(q^2)$, and we have that

$$\begin{aligned} 2q\delta_{q^2}(q) &= \delta_{q^2}(q^2) = \alpha(q^2) \\ &= \alpha(x_1^{2k(1)} + 2a x_1^{k(1)} x_2^{k(2)} \cdots x_n^{k(n)} + a^2 x_2^{2k(2)} \cdots x_n^{2k(n)}) \\ &= 2a\alpha(x_1^{k(1)} x_2^{k(2)} \cdots x_n^{k(n)}) \end{aligned}$$

from (1) and (2), hence it follows that

$$a^{-1} q \delta_{q^2}(q) = \alpha(x_1^{k(1)} \cdots x_n^{k(n)}). \tag{3}$$

With $x_1, \cdots, x_n$ replaced by $b_1, \cdots, b_n$, respectively, in (3), we have that

$$0 = \left[\alpha(x_1^{k(1)} \cdots x_n^{k(n)})\right]_{x_1 = b_1, \cdots, x_n = b_n},$$

for $q(b_1, \cdots, b_n) = 0$. Since $b_1, \cdots, b_n$ are arbitrary non-zero elements in $F$, $\alpha(x_1^{k(1)} \cdots x_n^{k(n)}) = 0$ in $F[X]$ ([6, Th.I.14]). It follows that $\alpha(x_1^{k(1)} \cdots x_n^{k(n)}) = 0$ for all monic monomial $x_1^{k(1)} \cdots x_n^{k(n)}$ in $F[x_1, \cdots, x_n]$, and hence $\alpha = 0$. □

Let $H_X$ be the free commutative monoid on $X$, $M$ an $F[X]$-module, and $\varphi$ a mapping of $F[X]$ into $M$. If $\delta_0 : H_X \to M$ is a mapping defined by

$$\begin{aligned} &\delta_0(x_1^{k(1)} \cdots x_n^{k(n)}) \\ &= \sum_{i=1}^{n} k(i) x_1^{k(1)} \cdots x_{i-1}^{k(i-1)} x_i^{k(i)-1} x_{i+1}^{k(i+1)} \cdots x_n^{k(n)} \varphi(x_i) \end{aligned}$$

for any $x_1^{k(1)} \cdots x_n^{k(n)} \in H_X$, then $\delta_0$ has a unique linear extension to a $F$-derivation $\delta : F[X] \to M$ given by

$$\delta\left(\sum_i r_i x_1^{k_i(1)} \cdots x_{n_i}^{k_i(n_i)}\right) = \sum_i r_i \delta_0(x_1^{k_i(1)} \cdots x_{n_i}^{k_i(n_i)}),$$

for $\sum_i r_i x_1^{k_i(1)} \cdots x_{n_i}^{k_i(n_i)} \in F[X]$, and $\delta(x^k) = \delta_0(x^k) = k x^{k-1} \varphi(x)$ for $x \in X$ and $k \in \mathbb{N}$, especially, $\delta(x) = \varphi(x)$, $x \in X$.

THEOREM 3.4. *Each local derivation of $F[x_1, \cdots, x_n]$ into $F[X]$ is a $F$-derivation, where $\{x_1, \cdots, x_n\}$ is a finite subset of $X$.*

*Proof.* Let $\alpha : F[x_1, \cdots, x_n] \to F[X]$ be a local derivation. If $n = 1$, the restriction of $\alpha$ to $F[x_j]$ $(j \in \{1, \cdots, n\})$ is a $F$-derivation from Theorem 3.2. Suppose that the restriction of $\alpha$ to $F[x_{j(1)}, \cdots, x_{j(r-1)}]$ is a $F$-derivation for all $(r-1)$-element subset $\{x_{j(1)}, \cdots, x_{j(r-1)}\}$ of $\{x_1, \cdots, x_n\}$ with $r \leq n$. We show that the same is true for all $r$-element subset of $\{x_1, \cdots, x_n\}$. It will suffice to prove that the restriction of $\alpha$ to $F[x_1, \cdots, x_r]$ is a $F$-derivation.

Let $H$ be the free commutative monoid on the set $\{x_1, \cdots, x_r\}$ and define a mapping $\delta_0 : H \to F[X]$ by

$$\begin{aligned}
&\delta_0(x_1^{k(1)} \cdots x_r^{k(r)}) \\
&= \sum_{i=1}^{r} k(i) x_1^{k(1)} \cdots x_{i-1}^{k(i-1)} x_i^{k(i)-1} x_{i+1}^{k(i+1)} \cdots x_r^{k(r)} \alpha(x_i),
\end{aligned}$$

where the $k(i)$ are non-negative integers. Then $\delta_0$ has a unique linear extension to a $F$-derivation $\delta$ of $F[x_1, \cdots, x_r]$ into $F[X]$ such that $\delta(x_i) = \alpha(x_i)$, $i \in \{1, \cdots, r\}$. Since $\delta$ and $\alpha$ are a $F$-derivation and a local derivation, respectively, $\alpha - \delta$ is a local derivation of $F[x_1, \cdots, x_r]$ into $F[X]$ such that $(\alpha - \delta)(x_i) = 0$, $i \in \{1, \cdots, r\}$. Since by assumption, the restriction of $\alpha$ to $F[x_{j(1)}, \cdots, x_{j(r-1)}]$ is a $F$-derivation for all $(r-1)$-element subset $\{x_{j(1)}, \cdots, x_{j(r-1)}\}$ of $\{x_1, \cdots, x_r\}(\subseteq \{x_1, \cdots, x_n\})$, $\alpha - \delta$ is a $F$-derivation of $F[x_{j(1)}, \cdots, x_{j(r-1)}]$ into $F[X]$. It follows that $\alpha - \delta = 0$ from Lemma 3.3, and $\alpha = \delta$ on $F[x_1, \cdots, x_r]$, whence $\alpha$ is a $F$-derivation of $F[x_1, \cdots, x_r]$ into $F[X]$. □

THEOREM 3.5. *Each local derivation on $F[X]$ is a $F$-derivation.*

*Proof.* Let $\alpha : F[X] \to F[X]$ be a local derivation and $H_X$ a free commutative monoid on $X$. Define a mapping $\delta_0 : H_X \to F[X]$ by

$$\begin{aligned}
&\delta_0(x_1^{k(1)} \cdots x_n^{k(n)}) \\
&= \sum_{i=1}^{n} k(i) x_1^{k(1)} \cdots x_{i-1}^{k(i-1)} x_i^{k(i)-1} x_{i+1}^{k(i+1)} \cdots x_n^{k(n)} \alpha(x_i),
\end{aligned}$$

for non-negative integers $k(i)$ $(i = 1, \cdots, n)$. Then $\delta_0$ has a unique extension to a $F$-derivation $\delta : F[X] \to F[X]$ such that $\delta(x) = \alpha(x)$ for all $x \in X$. Since $\alpha$ and $\delta$ are a local derivation and a $F$-derivation, respectively, $\alpha - \delta$ is a local derivation on $F[X]$.

Let $x_1^{k(1)} \cdots x_n^{k(n)}$ be a monic monomial in $F[X]$. Since the restriction of $\alpha$ to $F[x_1, \cdots, x_n]$ is a $F$-derivation from Theorem 3.4, the restriction of $\alpha - \delta$ to $F[x_1, \cdots, x_n]$ is also a $F$-derivation such that $(\alpha - \delta)(x_i) = 0$ for all $i \in \{1, \cdots, n\}$, and we have that

$$\begin{aligned}
&(\alpha - \delta)(x_1^{k(1)} \cdots x_n^{k(n)}) \\
&= \sum_{i=1}^{n} k(i) x_1^{k(1)} \cdots x_{i-1}^{k(i-1)} x_i^{k(i)-1} x_{i+1}^{k(i+1)} \cdots x_n^{k(n)} (\alpha - \delta)(x_i) \\
&= 0.
\end{aligned}$$

It follows that $(\alpha - \delta)(x_1^{k(1)} \cdots x_n^{k(n)}) = 0$ for all monic monomial $x_1^{k(1)} \cdots x_n^{k(n)} \in F[X]$, hence $\alpha - \delta = 0$ and $\alpha$ is a $F$-derivation. □

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