Bull. Korean Math. Soc. **36** (1999), No. 2, pp. 233–238

THE GENERALIZED WITT ALGEBRAS USING ADDITIVE MAPS I

KI-BONG NAM

ABSTRACT. Kawamoto generalized the Witt algebra using $F[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ instead of $F[x_1, \cdots, x_n]$. We construct the generalized Witt algebra $W_{g,h,n}$ by using additive mappings g, h from a set of integers into a field F of characteristic zero. We show that the Lie algebra $W_{g,h,n}$ is simple if g and h are injective, and also the Lie algebra $W_{g,h,n}$ has no ad-diagonalizable elements.

1. Introduction

Let F be a field of characteristic zero contains the set of integers Z. Let N be the set of non-negative integers. The Witt algebra is called the general algebra by Rudakov [8]. Kac [2] studied the generalized Witt algebra on the F-algebra in the formal power series $F[[x_1, \cdots, x_n]]$ for a fixed positive integer n. Nam [5] constructs the Lie algebra on the F-subalgebra $F[e^{\pm x_1}, \cdots, e^{\pm x_n}, x_1, \cdots, x_{n+1}, \cdots, x_{n+m}]$ in the formal power series $F[[x_1, \cdots, x_n]]$ for the given positive integers n and m.

Consider the generalized Witt algebra $W(n, m)$ having a basis

$$B = \{e^{a_1 x_1} \cdots e^{a_n x_n} x_1^{i_1} \cdots x_{n+m}^{i_{n+m}} \partial_k | a_1, \cdots, a_n, i_1, \cdots, i_{n+m} \in Z, \\ 1 \leq k \leq n+m\}$$

Received February 9, 1998.

1991 Mathematics Subject Classification: Primary 17B40, 17B65; Secondary 17B56, 17B68.

Key words and phrases: simple Lie algebras, Lie derivation, Lie automorphism.

with Lie bracket on basis elements given by

$$\begin{aligned}
&[e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_{n+m}^{i_{n+m}}\partial_l, e^{b_1x_1}\cdots e^{b_nx_n}x_1^{t_1}\cdots x_{n+m}^{t_{n+m}}\partial_j]\\
&= b_l e^{a_1x_1+b_1x_1}\cdots e^{a_nx_n+b_nx_n}x_1^{i_1+t_1}\cdots x_{n+m}^{i_{n+m}+t_{n+m}}\partial_j\\
&\quad + t_l e^{a_1x_1+b_1x_1}\cdots e^{a_nx_n+b_nx_n}x_1^{i_1+t_1}\cdots x_{n+m}^{i_{n+m}+t_{n+m}}x_l^{-1}\partial_j\\
&\quad - a_j e^{a_1x_1+b_1x_1}\cdots e^{a_nx_n+b_nx_n}x_1^{i_1+t_1}\cdots x_{n+m}^{i_{n+m}+t_{n+m}}\partial_l\\
&\quad - i_j e^{a_1x_1+b_1x_1}\cdots e^{a_nx_n+b_nx_n}x_1^{i_1+t_1}\cdots x_{n+m}^{i_{n+m}+t_{n+m}}x_j^{-1}\partial_l
\end{aligned}$$

where $b_i = 0$ if $n+1 \le i \le n+m$, and $a_j = 0$ if $n+1 \le j \le n+m$. See [2], [5], [6].
In [3], it is noted that the Lie subalgebra $W(0,m)$ of $W(n,m)$ is the Witt algebra on $F[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$.

Let g and h be additive maps from Z into F, where Z is the set of integers. Let us define the Lie algebra $W_{g,h,n}$ with basis

$$\left\{ \binom{a_1}{i_1}\cdots\binom{a_n}{i_n}_k | a_1,\cdots,a_n,i_1,\cdots,i_n \in Z, 1 \le k \le n \right\}$$

and a Lie bracket on basis elements given by

$$\begin{aligned}
&\left[\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}_k, \binom{b_1}{j_1}\cdots\binom{b_n}{j_n}_l\right]\\
&= g(b_k)\binom{a_1+b_1}{i_1+j_1}\cdots\binom{a_n+b_n}{i_n+j_n}_l\\
&\quad + h(j_k)\binom{a_1+b_1}{i_1+j_1}\cdots\binom{a_k+b_k}{i_k+j_k-1}\binom{a_{k+1}+b_{k+1}}{i_{k+1}+j_{k+1}}\cdots\binom{a_n+b_n}{i_n+j_n}_l\\
&\quad - g(a_l)\binom{a_1+b_1}{i_1+j_1}\cdots\binom{a_n+b_n}{i_n+j_n}_l\\
&\quad - h(i_l)\binom{a_1+b_1}{i_1+j_1}\cdots\binom{a_l+b_l}{i_l+j_l-1}\binom{a_{l+1}+b_{l+1}}{i_{l+1}+j_{l+1}}\cdots\binom{a_n+b_n}{i_n+j_n}_k.
\end{aligned}$$

We extend the above Lie bracket linearly to the given basis B (see [5], [6], [8].) Here it is not hard to show that the above bracket satisfies the Jacobi identity.

In this paper we will prove the following main theorem.

THEOREM. *The Lie algebra $W_{g,h,n}$ is simple.*

2. Main results

The Lie algebra $W_{g,h,n}$ has a Z^n-gradation as follows [3]:

$$W_{g,h,n} = \bigoplus_{(a_1,\cdots,a_n)\in Z^n} W_{(a_1,\cdots,a_n)}, \tag{1}$$

where $W_{(a_1,\cdots,a_n)}$ is the subspace of $W_{g,h,n}$ with basis

$$B = \left\{ \binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_k | a_1, \cdots, a_n, i_1, \cdots, i_n \in Z, 1 \leq k \leq n \right\}.$$

Let $W_{(a_1,\cdots,a_n)}$ denote the $(a_1, \cdots, a_n)$-homogeneous component of $W_{g,h,n}$ and call elements in $W_{(a_1,\cdots,a_n)}$ the $(a_1, \cdots, a_n)$-homogeneous elements. Note that the $(0, \cdots, 0)$-homogeneous component is isomorphic to the Witt algebra $W(n)$ [8]. From now on let the $(0, \cdots, 0)$-homogeneous component denote the 0-homogeneous component.

Let us define $H(l)$ to be the number of different homogeneous components for any $l \in W_{g,h,n}$.

For the simplicity of $W_{g,h,n}$, we assume the map g and g are injective maps. Now we introduce a lexicographic ordering on two basis elements of $W_{g,h,n}$ as follows:

For any two basis elements $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_l$ and $\binom{b_1}{j_1} \cdots \binom{b_n}{j_n}_k \in B$, we have $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_l > \binom{b_1}{j_1} \cdots \binom{b_n}{j_n}_k$ if

$$(b_1, \cdots, b_n, i_1, \cdots, j_n, k) \leq (a_1, \cdots, a_n, i_1, \cdots, i_n, l)$$

by the natural lexicographic ordering in $Z^{2n} \times Z$.

For any element $l \in W_{g,h,n}$, l can be written as follows using the ordering and the gradation:

$$l = \sum_{i_1,\cdots,i_n,p} C(i_1, \cdots, i_n, p) \binom{a_{11}}{i_1} \cdots \binom{a_{1n}}{i_n}_p + \cdots$$
$$+ \sum_{j_1,\cdots,j_n,q} C(j_1, \cdots, j_n, p) \binom{a_{t1}}{i_1} \cdots \binom{a_{tn}}{j_n}_q$$

where $(a_{11}, \cdots, a_{1n}) > \cdots > (a_{t1}, \cdots, a_{tn})$ using the natural ordering on Z^n.

Next, define the string number $st(l) = t$ for l (see [5], [6]), and $l_p(l)$ as the $\max\{i_1, \cdots, i_n, \cdots, j_1, \cdots, j_n\}$. For any basis element $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_l$ in B, let us call $a_1, \cdots, a_n$ the upper indices and $i_1, \cdots, i_n$ the lower indices.

REMARK. If g and h are inclusions, then we have $W_{g,h,n} = W(n, 0)$ where $W(n, 0)$ is the generalized Witt algebra studied by Kawamoto [3].

LEMMA 1. *If $l \in W_{g,h,n}$ is any non-zero element, then the ideal $\langle\ l\ \rangle$ generated by l contains an element whose lower indices are positive.*

Proof. Take an element $M = \binom{0}{j_1} \cdots \binom{0}{j_n}_l$ such that $j_1 >> \cdots >> j_n$ and t such that either $a_t \neq 0$ or $i_t \neq 0$ of l, where $a >> b$ means a is sufficiently larger than b. Then $0 \neq [M, l]$ is the required element. □

LEMMA 2. *If an ideal of $W_{g,h,n}$ contains $\binom{0}{0} \cdots \binom{0}{0}_i$ where $(1 \leq i \leq n)$, then we have $I = W_{g,h,n}$.*

Proof. The Witt algebra $W(n)$ is simple, so we know the ideal in the lemma contains $W(n)$ [3].

For any basis element $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t$ of $W_{g,h,n}$, if $i_1 = 0$ and $a_1 \neq 0$, then we have

$$\left[\binom{0}{0}_1, \binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t\right] = g(a_1)\binom{a_1}{i_1} \cdots \binom{a_n}{i_n}_t \in I.$$

We can assume for any fixed $p \in N$, $\binom{a_1}{p} \cdots \binom{a_n}{i_n}_t \in I$, that we have

$$\left[\binom{0}{0}_1, \binom{a_1}{p+1} \cdots \binom{a_n}{i_n}_t\right] - h(p+1)\binom{a_1}{p} \cdots \binom{a_n}{i_n}_t$$
$$= g(a_1)\binom{a_1}{p+1} \cdots \binom{a_n}{i_n}_t \in I$$

where $a_1 \neq 0$. We can assume for any fixed $p \in Z - N$ that

$$\left[\binom{0}{0}_1, \binom{a_1}{p} \cdots \binom{a_n}{i_n}_t\right] - g(a_1)\binom{a_1}{p} \cdots \binom{a_n}{i_n}_t$$
$$= h(p)\binom{a_1}{p-1} \cdots \binom{a_n}{i_n}_t \in I,$$

where $Z - N$ is the set of negative integers. Thus, we have $I = W_{g,h,n}$. Therefore, we have proven the lemma. □

THEOREM 1. *The Lie algebra $W_{g,h,n}$ is simple.*

Proof. Let I be any non-zero ideal of $W_{g,h,n}$. Let us prove this theorem by induction on $H(l)$ for any non-zero element of $l \in I$.

Let l be any nonzero element of I whose lower indices are positive integers by lemma 1. If $H(l) = 1$ and $l \in W_{(0,\cdots,0)} \cong W(n)$, then we have proved the theorem by lemma 2.

Assume that we have proved the theorem for any element $l \in I$ such that $H(l) = p$. Consider the element $l \in I$ such that $H(l) = p + 1$. If l contains a $(0, \cdots, 0)$-homogeneous component, then we have

$$0 \neq l_1 = \left[\binom{0}{0}_t, \left[\binom{0}{0}_t, \left[\cdots, \left[\binom{0}{0}_t, l\right] \cdots\right]\right]\right]$$

is the element which has no $(0, \cdots, 0)$-homogeneous elements by taking appropriate t. Then we have $H(l_1) < p + 1$, and the theorem is proved.

If l contains no $(0, \cdots, 0)$-homogeneous component, then l has an $(a_1, \cdots, a_n)$-homogeneous component. If we take an element $\binom{-a_1}{0} \cdots \binom{-a_n}{0}_t$ and take an appropriate t, then we have an element $l_2 = [\binom{-a_1}{0} \cdots \binom{-a_n}{0}_t, l] \neq 0$ such that $H(l_2) < p + 1$. Therefore, we have proven the theorem by induction and Lemma 2. □

COROLLARY 1. *The Lie algebra $W(n, 0)$ is simple.*

Proof. If we take an additive embedding $g, h : Z \to F$, then we get the required result (see [5], [6]). □

It is interesting problem to find all the automorphisms of $W_{g,I,1}$, where $I : Z \to F$ is an embedding.

THEOREM 2. *For any automorphism $\theta \in Aut(W_{g,I,1})$,*

$$\theta\left(\binom{0}{0}_1\right) = \sum_j C_j \binom{0}{j}_1,$$

where $C_j \in F$.

Proof. It is not difficult to prove this theorem using the gradation (1) of $W_{g,I,1}$ and the action of $\binom{0}{0}_1$ on the 0-homogeneous component W_0 whose vector space basis is $\{\binom{0}{l}_1 | l \in Z\}$ as an adjoint map. □

If the lower indices of $W_{g,I,1}$ are zero, then this is Block algebra [1], thus all the automorphisms of this Lie algebra are decided by Block in

Theorem 3 of [1]. The element $l \in W_{g,h,n}$ is ad-diagonalizable element if $[l, m] = \alpha(m)m$ for $m \in B$ and $\alpha(m) \in F$.

We have a following proposition.

PROPOSITION 1. *The Lie algebra $W_{g,h,n}$ has no non-zero ad-diagonalizable elements with respect to the basis B.*

Proof. Since $W_{g,h,n}$ is Z^n-graded Lie algebra, all the ad-diagonalizable elements are in the $(0, \cdots, 0)$-homogeneous component. The $(0, \cdots, 0)$-homogeneous component is isomorphic to the Witt algebra $W(n)$ [8]. Thus all the ad-diagonalizable elements of $W_{g,h,n}$ are of the form $\sum_{i=1}^{n} C_i \binom{0}{1}_i$ where $C_i \in F$. But $[\sum_{i=1}^{n} C_i \binom{0}{1}_i, \binom{a}{0}_j] \neq \alpha \binom{a}{0}_j$ for any $\alpha \in F$ where $a \neq 0$. Therefore, we have proved the proposition. □

REMARK. For the non-existence of ad-diagonalizable elements of $W(n, 0)$ see Corollary 1 of [5].

ACKNOWLEDGEMENT. The author thanks the referee for valuable suggestions on this paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706, USA
E-mail: nam@math.wisc.edu