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A CONSTRAINT ON SYMPLECTIC STRUCTURE OF $b_2^+ = 1$ MINIMAL SYMPLECTIC FOUR-MANIFOLD

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ABSTRACT. Let X be a minimal symplectic four-manifold with $b_2^+ = 1$ and $c_1(K)^2 \geq 0$. Then we show that there are no symplectic structures ω such that $c_1(K)\cdot\omega > 0$, if X contains an embedded symplectic submanifold Σ satisfying $\int_\Sigma c_1(K) < 0$.

1. Introduction

The results on the Seiberg-Witten invariants of four-manifolds have played important roles on studying symplectic structures of them. One of these is

THEOREM 1.1 (Taubes). *Let X be an oriented symplectic four manifold with $b_2^+ > 1$. Let ω be a symplectic form compatible with the orientation. Then $c_1(K^{\pm 1})$ on X has Seiberg-Witten invariant ± 1. (where $c_1(K)$ means the first Chern class of the canonical bundle associated with the almost complex structure on X).*

This result shows that four-manifolds whose Seiberg-Witten invariants do not take value ± 1 cannot have any symplectic structure.

THEOREM 1.2 (Taubes). *Let X be an oriented symplectic four-manifold with $b_2^+ > 1$. Let E be a nontrivial complex bundle over X and use E to define a $Spin^{\mathbf{C}}$-structure $L = det(S^+) \in Spin$ where $S^+ = E \oplus (K^{-1} \otimes E)$. Then $SW(L) = \pm Gr(c_1(E))$.*

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Here Taubes uses a new kind of Gromov invariant $Gr(V)$ counting embedded symplectic surfaces which represent the fundamental class of the Poincaré dual of the cohomology class $V \in H^2(X; \mathbb{Z})$.

Assuming $b_2^+ > 1$ Taubes' above result says that $c_1(K)$ has a nonzero Gromov invariant and that the Poincaré dual of $c_1(K) \in H^2(X; \mathbb{Z})$ is represented by a smooth symplectic curve which may not be connected.

Taubes also studies symplectic 4-manifolds with $b_2^+ = 1$, in which the Seiberg-Witten invariants depend on the metrics of the manifolds. In particular he shows that there is no symplectic ω on $\mathbb{CP}^2$ with $c_1(K) \cdot \omega > 0$, and that $c_1(K) \cdot \omega < 0$ for the standard symplectic structure ω on $\mathbb{CP}^2$. By McDuff's theorem on the intersection of symplectic curves, the intersection number of Poincaré dual to $c_1(K)$ and a symplectic curve in X is non-negative. So one could easily conclude that if a symplectic four-manifold contains a symplectically embedded curve Σ with $c_1(K) \cdot [\Sigma] < 0$, then its b_2^+ must be one.

In this note we would like to extend Taubes' result on $\mathbb{CP}^2$ to minimal symplectic four-manifolds containing an embedded symplectic curve Σ with $c_1(K) \cdot [\Sigma] < 0$.

2. Wall-crossing

For a four-manifold X with $b_2^+ = 1$, the Seiberg-Witten invariant is no longer a smooth invariant of the underlying manifold, because we can not avoid reducible solutions when we deform metrics on X. The moduli space has singularities at reducible solutions, and the invariant may jump. Given a metric g, there is a unique associated self-dual harmonic 2-form ω_g for g, mod nonzero scalars. For Seiberg-witten equation, a reducible solution exists if and only if $P_+ F_A = 0$, that is $c_1(L) \cdot \omega_g = 0$.

Recall the Seiberg-Witten, SW, equations in the Taubes' construction.

Standard SW equations:

$$D_A \psi = 0,$$

$$P_+ F_A = \frac{1}{4} \tau(\psi \otimes \psi^*).$$

Perturbed SW equations:

$$D_A \psi = 0,$$

$$P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + tP_+F_{A_0} - \frac{it}{4}\omega,$$

where $0 \leq t \leq 1$ and A_0 is a canonical connection on K^{-1} (up to gauge equivalence).

Deformed SW equations:

$$D_A\psi = 0,$$

$$P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + P_+F_{A_0} - \frac{ir}{4}\omega,$$

where $r \geq 1$ is a real parameter.

A wall can appear in any of following three types.

Type 1. The standard metric wall: where anti-self-dual harmonic 2-form suddenly appear.

Type 2. From the standard SW equation to the perturbed SW equation: there might be some wall for $t \in [0, 1]$.

Type 3. From the perturbed SW equation to the deformed SW equation: there might be some wall for $r \geq 1$.

Let us recall some results in [8] for a Spin ${}^{\mathbb{C}}$-structure $S^+ = E \oplus (K^{-1} \otimes E)$ on a symplectic manifold (X, ω) with $b_2^+ = 1$.

LEMMA 2.1. [8] *If $c_1(E) \cdot \omega \leq 0$ (in particular, if $c_1(E) \cdot \omega = 0$), then there are no walls in type 3.*

Proof. Suppose that $(A, 0)$ occurs as a reducible solution. Wedge ω on the both sides of the deformed equation

$$P_+F_A = P_+F_{A_0} - \frac{ir}{4}\omega,$$

for $r \geq 1$ and integrate over X, then we get

$$8\pi c_1(K^{-1} \otimes E^2) \cdot \omega = 8\pi c_1(K^{-1}) \cdot \omega + r\omega \cdot \omega.$$

So, $16\pi c_1(E) \cdot \omega = r\omega \cdot \omega$. But since we assume that $c_1(E) \cdot \omega \leq 0$, any r in the region $r \geq 0$ does not satisfy the above equation. □

LEMMA 2.2. [8] *If $c_1(K^{-1} + 2E) \cdot \omega > 0$, then there is an odd number of walls in type 2 and 3. If $c_1(K^{-1} + 2E) \cdot \omega < 0$, then there is an even number of walls in type 2 and 3.*

3. Minimal symplectic four-manifolds with $b_2^+ = 1$

In this section we would like to introduce some results on minimal symplectic four-manifolds with $b_2^+ = 1$.

THEOREM 3.1 (McDuff). *If a symplectic four-manifold X has a non-negative self-intersecting rational curve, then it must be symplectomorphic to either rational, rational ruled, or irrational ruled manifolds.*

LEMMA 3.2. *Let $c_1(K)$ be an element in $H^2(X;\mathbb{Z})$ such that $c_1(K)^2 < 0$. Then there exists an integral element Z of the forward light cone such that $Z \cdot Z = 0$ and $c_1(K) \cdot Z < 0$.*

If a minimal symplectic 4-manifold X with $c_1(K)^2 < 0$ is $b_1(X) = 0$, using the adjunction formula and McDuff's Theorem 3.1, we can show that $SW(K^{-1} + 2kZ) = Gr(kZ)$ is zero identically for every positive integer k. Here Z is an integral class defined in Lemma 3.2. For sufficiently large k,

$$c_1(K^{-1} + 2kZ) \cdot \omega > 0,$$

and by the Lemma 2.2, the number of walls crossed in two types is odd. If we deform from Taubes' chamber, we have nonzero SW invariants for sufficiently large k (by using the wall crossing formula in the case of $b_1 = 0$). But this contradicts to the finiteness of basic classes. From this fact we may have that

PROPOSITION 3.3. [6] *If X is a minimal symplectic four-manifold with $c_1(K)^2 < 0$, then its first Betti number b_1 must be nonzero.*

LEMMA 3.4. [6] *Let X be a minimal symplectic four-manifold with $c_1(K)^2 < 0$. Then for all nonzero $y \in H^1(X;\mathbb{R})$, the map*

$$y\cup : H^1(X;\mathbb{R}) \to H^2(X;\mathbb{R})$$

must be nonzero and there exists an integral basis of $H^1(X;\mathbb{R})$ which is nondegenerate.

Let C be a nonzero image of the cup product $\cup : H^1 \times H^1 \to H^2$.
If we choose C in the forward light cone, we know that

COROLLARY 3.5. [6] *A $Spin^{\mathbb{C}}$-structure L (with nonnegative moduli space dimension) has a non-zero wall crossing if $c_1(L) \cdot C \neq 0$.*

Suppose that X is not an irrational ruled surface. As in the Proposition 3.3, we can show that $Gr(kZ) = 0$ for all classes $k \cdot Z$ and the number of walls crossed in the type 2 and 3 is odd for sufficiently large k. Then by the finiteness of Seiberg-Witten basic classes and by Corollary 3.5, we have $c_1(K^{-1} + 2kZ) \cdot C = 0$ for sufficiently large k. So,

$$c_1(K) \cdot C = 0, \quad Z \cdot C = 0$$

and from the fact that for any nonzero elements a, b in the closure of the forward cone, $a \cdot b = 0$ if and only if $b = \gamma a$ for some $\gamma > 0$, we get

$$C = \alpha Z \quad (\alpha \neq 0).$$

This contradicts to $c_1(K) \cdot Z < 0$. Therefore we get

THEOREM 3.6. [6] *Let X be a minimal symplectic four manifold with $c_1(K)^2 < 0$. Then X must be irrational ruled.*

Also we could see from [6] that

THEOREM 3.7. *Let X be a symplectic four-manifold $b_2^+ = 1$. If $c_1(K) \cdot \omega < 0$, then it must be either rational, rational ruled, or irrational ruled.*

4. Main theorem

We are now ready to prove our main theorem:

THEOREM 4.1. *Let X be a minimal symplectic four-manifold with $b_2^+ = 1$ and $c_1(K)^2 \geq 0$. Let Σ be an embedded symplectic 2-dimensional submanifold satisfying $c_1(K) \cdot \Sigma < 0$. Then there is no symplectic structure ω on X such that $c_1(K) \cdot \omega > 0$.*

First we consider a wall in type 1.

LEMMA 4.2. *Let X be a $b_2^+ = 1$ symplectic four-manifold. If $c_1(K)^2 \geq 0$ and $c_1(K) \cdot \omega > 0$, then there are no walls in type 1.*

Proof. Suppose that there is a wall in type 1. Then there is a unique self-dual harmonic 2-form ω' of norm one so that for a $L \in H^2(X; \mathbb{Z})$ with $L^2 \geq K^2 \geq 0$, $c_1(L) \cdot \omega' = 0$. The quadratic form can be diagonalized in a real basis. The form of coordinate system $(x, y_1, y_2, \cdots, y_n)$ is represented as $x^2 - \sum_{i=1}^n y_i^2$. If $c_1(L) = (x, y_1, y_2, \cdots, y_n)$ and $\omega' = (a, b_1, b_2, \cdots, b_n)$, then

$$c_1(L)^2 = x^2 - \sum_{i=1}^n y_i^2 \geq K^2 \geq 0, \qquad (\omega')^2 = a^2 - \sum_{i=1}^n b_i^2 = 1$$

and

$$c_1(L) \cdot \omega' = xa - \sum_{i=1}^n y_i b_i = 0.$$

If we let $a = \sum_{i=1}^n (y_i/x) b_i$, then the Cauchy-Schwartz inequality implies

$$a^2 = \left(\sum_{i=1}^n \frac{y_i}{x} b_i \right)^2 \leq \left(\sum_{i=1}^n \frac{y_i^2}{x^2} \right) \left(\sum_{i=1}^n b_i^2 \right) \leq \sum_{i=1}^n b_i^2$$

since $x^2 - \sum_{i=1}^n y_i^2 \geq 0$. But $a^2 = 1 + \sum_{i=1}^n b_i^2$ leads to contradiction. □

Second we consider a wall in type 2.

LEMMA 4.3. *If $c_1(K) \cdot \omega > 0$, then there are no walls in type 2.*

Proof. By Lemma 2.2, since $c_1(K^{-1}) \cdot \omega < 0$, there is an even number of reducible solutions to the equations

$$D_A \psi = 0, \quad F_A^+ = \frac{1}{4}\tau(\psi \otimes \psi^*) + tF_{A_0}^+ - \frac{it}{4}\omega, \quad 0 \leq t \leq 1.$$

Wedge ω on the both sides of the perturbed SW equations and integrate over X, then we have

$$8\pi c_1(K^{-1}) \cdot \omega = 8\pi t c_1(K^{-1}) \cdot \omega + t\omega \cdot \omega.$$

So

$$8\pi(1-t)c_1(K^{-1}) \cdot \omega = t\omega \cdot \omega.$$

If $c_1(K^{-1}) \cdot \omega < 0$, then it is impossible. □

Finally by Lemma 2.1, if $c_1(E) \cdot \omega = 0$, then there are no walls in type 3.

Also we consider following theorem.

THEOREM 4.4 (Taubes). *Let X be a compact, oriented 4-manifold with $b_2^+ = 1$ and with a symplectic form. Then the symplectic form canonically defines a chamber in which the equivalence $SW = Gr$ holds for classes $e \in H^2(X; \mathbb{Z})$ which obey $\langle e, s \rangle \geq -1$ wherever $s \in H_2(X; \mathbb{Z})$*

is represented by an embedded, symplectic sphere with self-intersection number -1.

This theorem is proved in [13]. Now we prove Theorem 4.1

Proof of Theorem 4.1 Suppose that there is a symplectic structure ω such that $c_1(K) \cdot \omega > 0$. Then by Lemma 4.2 and Lemma 4.3, there are no walls. Hence $SW(K)$ is an invariant independent of the metric and its value $SW(K) = \pm 1$. Furthermore since we assumed that X is minimal, Taubes' theorem $SW(K) = \pm Gr(c_1(K))$ holds in a chamber and we conclude that $SW(K) = \pm Gr(c_1(K))$ is not equal to zero in the chamber. So the Poincaré dual of $c_1(K)$ is represented by a symplectic curve and then $c_1(K) \cdot [\Sigma] \geq 0$ for a symplectic submanifold Σ on X. This contradicts to our assumption. □

5. Some examples

If we apply Theorem 4.1 to the complex projective plane $\mathbb{CP}^2$, then we have

THEOREM 5.1 (Taubes). *The manifold* $\mathbb{CP}^2$ *has no symplectic form* ω *for which* $c_1(K) \cdot [\omega] > 0$. *(The standard Kähler structure on* $\mathbb{CP}^2$ *has* $c_1(K) \cdot [\omega] < 0$.*)*

Let X be an S^2-bundle over a Riemannian surface Σ with genus $g(\Sigma) = 0$ or 1. Let $c_1(K) = ax + by$ and $\omega = cx + dy$, where x and y represent the base class and fiber class, respectively. Then $c_1(K)^2 = 2\chi + 3\sigma \geq 0$ and so that $c_1(K)^2 = 2ab \geq 0$ if X is a trivial bundle. We know that the fiber class y is represented by an embedded rational curve which cannot be decomposed into a disjoint union of embedded smooth submanifolds. Let $c_1(K) \cdot y = a < 0$, then $b \leq 0$. From Theorem 4.1, $c_1(K) \cdot \omega = (ax + by) \cdot (cx + dy) = bc + ad \leq 0$ with $\omega^2 = 2cd > 0$. Then $c > 0$ and $d > 0$.

If X is nontrivial S^2-bundle, then $c_1(K)^2 = a^2 + 2ab \geq 0$ and $a + 2b \leq 0$ if $a < 0$. Then $c_1(K) \cdot \omega = ac + bc + ad \leq 0$ where $\omega^2 = c^2 + 2cd = c(c+2d) > 0$. Since $2c_1(K) \cdot \omega = 2(ad + ac + bc) = a(c+2d) + c(a+2b) < 0$. Then $c > 0$ and $c + 2d > 0$.

PROPOSITION 5.2. [8] Let X be a ruled surface over a Riemannian surface Σ with genus $g(\Sigma) = 0$ or 1. Let $c_1(K) = ax + by$ and $\omega = cx + dy$, where $a < 0$.

(i) If X is trivial, then $c > 0$ and $d > 0$.

(ii) If X is nontrivial, then $c > 0$ and $c + 2d > 0$.

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