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INVARIANT OPEN SETS UNDER COCOMPACT AFFINE ACTIONS

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ABSTRACT. In this paper, we find a condition on an open subset of the affine space which admits a cocompact affine action. To do it, the asymptotic flag of an open convex subset is introduced and some applications to affine manifolds are presented.

1. Introduction

An affine manifold is a manifold locally modeled on the affine space $\mathbb{E}^n$. A chart map is analytically continued to define a developing map on the universal covering space into the affine space. A deck transformation is transferred to an affine transformation via the developing map, which is called a holonomy homomorphism.

To study affine actions is useful in the theory of affine manifolds. We prove that an open subset of the affine space which lies between two parallel hyperplane cannot admit a cocompact affine action. To get the result, we introduce the notion of the asymptotic flag of an open convex subset of $\mathbb{E}^n$. The asymptotic flag of an open convex subset C is a flag

$$\{O\} \subset \text{Asymp}_1(C) \subset \cdots \subset \text{Asymp}_k(C) = \mathbb{R}^n$$

in the underlying vector space $\mathbb{R}^n$. $\text{Asymp}_1(C)$ is a subspace spanned, roughly, by infinite directions of C, and $\text{Asymp}_i(C) = \pi_{i-1}^{-1}(\text{Asymp}_1(q_{i-1}(C)))$ where $\pi_{i-1} : \mathbb{R}^n \to \mathbb{R}^n/\text{Asymp}_{i-1}(C)$ and $q_{i-1} : \mathbb{E}^n \to \mathbb{E}^n/\text{Asymp}_{i-1}(C)$. The asymptotic flag is invariant under any affine action on C.

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Our result is a generalization of the Theorem of D. Fried [2]. J. Milnor proposed a conjecture on the Euler characteristic of affine manifolds. D. Fried partially answered this conjecture, which is a generalization of the result of F. Kamber and P. Tondeur, and we reprove it at the end of this paper.

2. The asymptotic flag

Let C be an open convex subset of the affine space $\mathbb{E}^n$. A nonzero vector $v \in \mathbb{R}^n$ is called an *asymptotic vector* of C at x if $x + tv$ is contained in C for $t \geq 0$. The convexity implies that the definition of an asymptotic vector does not depend on the choice of a base point x.

Let $\mathrm{Asymp}_1(C)$ be the linear subspace of the underlying vector space $\mathbb{R}^n$ spanned by asymptotic vectors, which is called the *asymptotic subspace* associated with C. For convenience, we define by $\mathbb{R}^n$ the asymptotic subspace associated with a bounded convex subset. Suppose that $\mathrm{Asymp}_i(C) \subsetneq \mathbb{R}^n$ is defined. Let $q_i : \mathbb{E}^n \to \mathbb{E}^n/\mathrm{Asymp}_i(C)$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}^n/\mathrm{Asymp}_i(C)$ be quotient maps. Since $q_i(C)$ is open and convex, $\mathrm{Asymp}_1(q_i(C)) \subset \mathbb{R}^n/\mathrm{Asymp}_i(C)$ is defined. We define $\mathrm{Asymp}_{i+1}(C) = \pi_i^{-1}(\mathrm{Asymp}_1(q_i(C))) \subset \mathbb{R}^n$. Applying this process repeatedly whenever $\mathrm{Asymp}_i(C) \neq \mathbb{R}^n$, we obtain a flag

$$\{O\} \subset \mathrm{Asymp}_1(C) \subset \mathrm{Asymp}_2(C) \subset \cdots \subset \mathrm{Asymp}_k(C) = \mathbb{R}^n$$

which is called the *asymptotic flag* associated with C. Either the convex set $q_{k-1}(C)$ is bounded or $\mathbb{R}^n/\mathrm{Asymp}_{k-1}(C)$ itself is spanned by asymptotic vectors of $q_{k-1}(C)$. C is said to be *eventually bounded* if the former holds.

The following lemma is obvious.

LEMMA 2.1. *An open convex set C is eventually bounded if and only if it lies between two parallel hyperplanes.*

Now we consider an affine action on C. Suppose that Γ is a subgroup of $\mathrm{Aff}(n)$ which leaves C invariant. For $\gamma = (a, A) \in \Gamma$ and for an asymptotic vector v, $\gamma(x + tv) = \gamma(x) + tAv$. This implies that

$$\gamma(x + \mathrm{Asymp}_1(C)) = \gamma(x) + \mathrm{Asymp}_1(C).$$

Similarly, we obtain $\gamma(x+\mathrm{Asymp}_i(C)) = \gamma(x) + \mathrm{Asymp}_i(C)$ for $i = 1, \dots, k$. Therefore the linear part of γ preserves the asymptotic flag associated with C. We choose an ordered basis of $\mathbb{R}^n$ so that every element of Γ is represented by the following form:

$$(2.2) \qquad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}, \quad \begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix}$$

where each a_i is an n_i-dimensional vector and each A_i is a square matrix of rank n_i and $n_i = \dim \mathrm{Asymp}_i(C) - \dim \mathrm{Asymp}_{i-1}(C)$.

If C is eventually bounded, then $q_{k-1}(C)$ is bounded. We choose an origin O so that $q_{k-1}(O)$ is the center of $q_{k-1}(C)$. There is a basis of $\mathbb{R}^n/\mathrm{Asymp}_{k-1}(C)$ such that the subgroup of $\mathrm{Aff}(\mathbb{E}^n/\mathrm{Asymp}_{k-1}(C))$ which leaves $q_{k-1}(C)$ invariant is orthogonal. Hence any element in Γ is represented by the form (2.2) with additional conditions:

$$(2.3) \qquad a_k = 0, \qquad A_k \in \mathrm{O}(\dim \mathbb{R}^n/\mathrm{Asymp}_{k-1}(C)).$$

Here we state and prove our main theorem.

THEOREM 2.4. *Let Γ be an affine group. Suppose that X is an Γ invariant open subset of $\mathbb{E}^n$. If $\Gamma\backslash X$ is compact, then X cannot lie between two parallel hyperplanes.*

Proof. Assume to contrary that X lies between two parallel hyperplanes. Let C be the convex hull of X. Then C also lies between two parallel hyperplanes, that is, it is eventually bounded. Let

$$\{O\} \subset \mathrm{Asymp}_1(C) \subset \cdots \subset \mathrm{Asymp}_k(C) = \mathbb{R}^n$$

be the asymptotic flag associated with C. We choose an origin $O \in \mathbb{E}^n$, a basis of $\mathbb{R}^n$ so that every element of Γ is of the form (2.3). Let $F \subset X$ be a compact subset which satisfies $\Gamma F = X$. Let $f : X \to \mathbb{R}$ be the map defined by

$$f(x) = \min\{\|x-y\| \mid y \in O + \mathrm{Asymp}_{k-1}(C)\}$$

where $\|\cdot\|$ is the Euclidean distance with respect to given basis. Let $x_0 \in F$ be a point which realizes the maximum of $f|_F$. In virtue of (2.3) f is Γ-invariant and hence f has the maximum at x_0. This implies that x_0 is a boundary point of the open set X. It is absurd. □

3. Applications to affine manifolds

In this section, we apply Theorem 2.4 to the theory of affine manifolds. Over all this section, M is a closed affine manifold and $\tilde{M}$ the universal covering space of M. $D : \tilde{M} \to M$ is a developing map and $\rho : \pi_1(M) \to \mathrm{Aff}(n)$ a holonomy homomorphism. The following Corollary is a generalization of the result of D. Fried [2].

COROLLARY 3.1. *Let M be a closed affine manifold. Then any open subset which is invariant under the holonomy action cannot lie between two parallel hyperplanes.*

In particular, a developing image cannot lie between two parallel hyperplanes.

Proof. Assume that there is a holonomy invariant open subset which lies between two parallel hyperplanes. We may assume that every element of the holonomy group Γ satisfies the condition (2.3). There is a compact subset $F \subset \tilde{M}$ such that $D(\tilde{M}) = \Gamma D(F)$. Hence $D(\tilde{M})$ lies between two parallel hyperplanes. Since $\Gamma \backslash D(\tilde{M})$ is compact, it is a contradiction. □

As an another application, we prove that some class of affine manifolds are of Euler characteristic zero. The result was already proved by D. Fried. We reprove it. In the book of M. Berger [Be, 11.3.8], he characterized the boundary of an open convex set.

PROPOSITION 3.2. *Let C be a convex subset of $\mathbb{E}^n$ with nonempty interior. If ∂C is not empty, then ∂C is either homeomorphic to $\mathbb{E}^{n-1}$ or affinely equivalent to $\mathbb{E}^r \times S$, where S is homeomorphic to $\mathbb{S}^{n-r-1}$, $(0 \leq r \leq n-1)$.*

The bundle $\tilde{M} \times_\rho \mathbb{E}^n$ is isomorphic to the tangent bundle of M([3]). Moreover the section $m \mapsto [\tilde{m}, D(\tilde{m})]$ corresponds to the zero section. Combining (3.1) and (3.2), we obtain

COROLLARY 3.3([4], [2]). *Let M be a closed affine manifold. If the convex hull of $D(\tilde{M})$ is a proper subset of $\mathbb{E}^n$, then the Euler characteristic of M vanishes.*

Proof. Let C be the convex hull of $D(\tilde{M})$. If C is a proper subset of $\mathbb{E}^n$, then the boundary ∂C is homeomorphic to either $\mathbb{E}^{n-1}$ or $\mathbb{E}^r \times \mathbb{S}^{n-r-1}$, $(0 \leq r \leq n-1)$. If the latter held, then $D(\tilde{M})$ would be bounded by two parallel hyperplanes. Hence the former holds and ∂C is contractible. The subbundle $\tilde{M} \times_\rho \partial C$ of $\tilde{M} \times_\rho \mathbb{E}^n$ avoids from the section $[\tilde{m}, D(\tilde{m})]$. Therefore the tangent bundle of M admits a nowhere vanishing section. □

References

[1] M. Berger, *Geometry I*, Springer-Verlag, 1987.

[2] D. Fried, *Polynomials on affine manifolds*, Trans. Amer. Math. Soc. **274** (1982), 709–719.

[3] W. Goldman and M. Hirsch, *The Radiance Obstruction and Parallel Forms on Affine Manifolds*, Trans. Amer. Math. Soc. **286** (1984), 629–649.

[4] F. Kamber and P. Tondeur, *Flat manifolds*, Lecture Notes in Mathematics 67, Springer-Verlag, 1968.

[5] John Milnor, *On the existence of a connection with curvature zero*, Comm. Math. Helv. **32** (1957), 215–223.

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