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IMBEDDINGS OF MANIFOLDS DEFINED ON AN O-MINIMAL STRUCTURE ON $(\mathbb{R}, +, \cdot, <)$

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ABSTRACT. Let M be an o-minimal structure on the standard structure $\mathfrak{R} := (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. We study $C^r\mathfrak{S}$ manifolds and $C^r\mathfrak{S}$-G manifolds $(0 \leq r \leq \omega)$ which are generalizations of Nash manifolds and Nash G manifolds. We prove that if M is polynomially bounded, then every $C^r\mathfrak{S}$ $(0 \leq r < \infty)$ manifold is $C^r\mathfrak{S}$ imbeddable into some $\mathbb{R}^n$, and that if M is exponential and G is a compact affine $C^\omega\mathfrak{S}$ group, then each compact $C^\infty\mathfrak{S}$-G manifold is $C^\infty\mathfrak{S}$-G imbeddable into some representation of G.

1. Introduction

M. Shiota [15] proved that every C^r $(r < \infty)$ Nash manifold is C^r Nash imbeddable into some $\mathbb{R}^n$, and that for any compact or compactifiable C^∞ manifold X of positive dimension, there exists a uncountable family of $\{Y_\lambda\}_{\lambda \in \Lambda}$ of nonaffine Nash manifolds such that each Y_λ is C^∞ diffeomorphic to X and that Y_λ is not Nash diffeomorphic to Y_μ for $\lambda \neq \mu$. Here a C^∞ manifold is *compactifiable* if it is C^∞ diffeomorphic to the interior of some compact C^∞ manifold with boundary, and a Nash manifold is *affine* if it admits a Nash imbedding into some $\mathbb{R}^n$.

There are results on equivariant generalizations of Nash manifolds [7], [9].

In the present paper we are concerned with generalizations of C^r $(r < \infty)$ Nash imbeddings of C^r $(r < \infty)$ Nash manifolds in an o-minimal structure on the standard structure $\mathfrak{R} := (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. O-minimal structures have desirable properties, and some of good references of them are [2], [3].

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In this paper M denotes an o-minimal structure on $\mathfrak{R}$, a subset of $\mathbb{R}^n$ definable in a structure M means it is definable in M with parameters, $\mathfrak{S}$ stands for the collection of all subsets of $\mathbb{R}^n$ ($n \in \mathbb{N}$) definable in M, an $\mathfrak{S}$ set means an element of $\mathfrak{S}$, and all manifolds do not have boundaries unless otherwise stated.

Let r be a non-negative integer, ∞ or ω. A C^r manifold X is called a $C^r\mathfrak{S}$ *manifold* if it admits a finite system of charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^k\}$ such that each gluing map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is a $C^r\mathfrak{S}$ diffeomorphism (an $\mathfrak{S}$ homeomorphism if $r = 0$), where k denotes the dimension of X. We say that a $C^r\mathfrak{S}$ manifold is *affine* if it is $C^r\mathfrak{S}$ imbeddable into some $\mathbb{R}^n$. If $M = \mathfrak{R}$ (resp. $\mathbf{R}_{\exp} = (\mathbb{R}, +, \cdot, <, \exp)$), then a $C^\omega\mathfrak{S}$ manifold is called a Nash manifold [15] (resp. an exponentially Nash manifold [8]). Remark that the family of Nash manifolds is the smallest family of $C^\omega\mathfrak{S}$ manifolds.

We call M *polynomially bounded* if for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in M, there exist an integer N and a real number x_0 such that $|f(x)| \le x^N$ for any $x > x_0$. Otherwise, M is called *exponential* by a result of C. Miller [12]. Notice that $\mathfrak{R}$ is polynomially bounded. If M is exponential, then the smallest family of $C^\omega\mathfrak{S}$ manifolds is the family of exponentially Nash manifolds.

THEOREM 1.1. *Let M be polynomially bounded and let r be a non-negative integer. Then every $C^r\mathfrak{S}$ manifold is affine.*

Remark that if $r = \infty$, then Theorem 1.1 is not true because there exist uncountably many compact nonaffine Nash manifolds when $M = \mathfrak{R}$ [15] and every C^∞ Nash diffeomorphism is a C^ω Nash diffeomorphism [11].

By a way similar to define Nash G manifolds and affine Nash G manifolds when G is a Nash group, we can define $C^r\mathfrak{S}$-G ($0 \le r \le \omega$) manifolds and affine $C^r\mathfrak{S}$-G ($0 \le r \le \omega$) manifolds when G is a $C^r\mathfrak{S}$ group (See Definition 2.4 and 2.5).

THEOREM 1.2. *Let M be exponential and let G be a compact affine $C^\infty\mathfrak{S}$ group. Then every compact $C^\infty\mathfrak{S}$-G manifold is affine.*

Theorem 1.2 is an o-minimal version of a result of R. S. Palais [13].

In section 2 we state preliminary results. We prove our results in section 3 and 4.

2. $C^r\mathfrak{S}$ manifolds and $C^r\mathfrak{S}$-G manifolds

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be $\mathfrak{S}$ sets. A map $f : X \longrightarrow Y$ is called an $\mathfrak{S}$ *map* if the graph of f ($\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m$) is an $\mathfrak{S}$ set.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open $\mathfrak{S}$ sets and let r be a non-negative integer, ∞ or ω. A C^r map $f : U \longrightarrow V$ is called a $C^r\mathfrak{S}$ *map* if it is an $\mathfrak{S}$ map. A $C^r\mathfrak{S}$ map $h : U \longrightarrow V$ is called a $C^r\mathfrak{S}$ *diffeomorphism* (an $\mathfrak{S}$ *homeomorphism* if $r = 0$) if there exists a $C^r\mathfrak{S}$ map $k : V \longrightarrow U$ such that $h \circ k = id$ and $k \circ h = id$.

THEOREM 2.1 (cf. [3]). *Let $S_1, \cdots, S_k \subset \mathbb{R}^n$ be $\mathfrak{S}$ sets and let r be a positive integer.*
(1) (*Cell decomposition*) *There exists a C^r cell decomposition of $\mathbb{R}^n$ compatible with $\{S_1, \cdots, S_k\}$.*
(2) (*Whitney stratification*) *There exists a finite C^r Whitney stratification of $\mathbb{R}^n$ compatible with $\{S_1, \cdots, S_k\}$, with each stratum a C^r cell in $\mathbb{R}^n$.*
(3) (*Triangulation*) *Let $S \subset \mathbb{R}^n$ be an $\mathfrak{S}$ set with $S_1, \cdots, S_k \subset S$. Then there exist a finite simplicial complex K in $\mathbb{R}^n$ and an $\mathfrak{S}$ map $\phi : S \longrightarrow \mathbb{R}^n$ such that ϕ maps S and each S_i homeomorphically onto a union of open simplexes of K.*

Theorem 2.1 allows us to define the *dimension* of an $\mathfrak{S}$ subset X of $\mathbb{R}^n$ by

$$\dim X = \max\{\dim \Gamma | \Gamma \text{ is a } C^r \text{ submanifold of } \mathbb{R}^n \text{ contained in } X\}.$$

One can easily see that the above dimension is well-defined.

EXAMPLE 2.2. (1) Let M be exponential. Then the C^∞ function $F : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

is a $C^\infty\mathfrak{S}$ function but not a C^ω function. Such a function does not exist in the usual Nash category. Recall that every C^∞ Nash map is a C^ω Nash map [11].

(2) It is well-known that the dimension of a semialgebraic set coincides with that of its Zariski closure (cf. [1]). But the Zariski closure of the graph of the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathbb{R}^2$ is the whole space $\mathbb{R}^2$. Hence if M is exponential, then in general the dimension of an $\mathfrak{S}$ set and that of its Zariski closure do not coincide.
(3) A non-constant periodic function $\mathbb{R} \longrightarrow \mathbb{R}$ is not an $\mathfrak{S}$ function in any M.

Let r be a non-negative integer, ∞ or ω. We can define $C^r\mathfrak{S}$ submanifolds of $\mathbb{R}^n$, $C^r\mathfrak{S}$ manifolds, $C^r\mathfrak{S}$ maps, $C^r\mathfrak{S}$ diffeomorphisms, and affine $C^r\mathfrak{S}$ manifolds as well as Nash ones (cf. [16]). Remark that the inverse function theorem is true in the $C^r\mathfrak{S}$ $(r > 0)$ category.

Notice that we do not assume that a $C^r\mathfrak{S}$ submanifold of $\mathbb{R}^n$ admits a finite family of charts. By Theorem 2.1 (3) and the proof of I.3.9 [16], a $C^r\mathfrak{S}$ $(r > 0)$ submanifold of $\mathbb{R}^n$ admits such a family, therefore it is of course a $C^r\mathfrak{S}$ manifold.

It is known that there exists a nonaffine Nash manifold [15], and that there exists a nonaffine exponentially Nash manifold [8]. However we do not know whether there exists a nonaffine $C^\omega\mathfrak{S}$ manifold for any M or not.

PROPOSITION 2.3 (cf. [3]). *If X is a nonempty $\mathfrak{S}$ subset of $\mathbb{R}^n$, then* $\dim \overline{X} = \dim X$ *and* $\dim X > \dim(\overline{X} - X)$, *where* $\dim \emptyset = -\infty$.

We define some notions to consider $C^r\mathfrak{S}$-G manifolds.

DEFINITION 2.4. Let r be a non-negative integer, ∞ or ω.
(1) A group G is called a $C^r\mathfrak{S}$ *group* (resp. an *affine* $C^r\mathfrak{S}$ *group*) if G is a $C^r\mathfrak{S}$ manifold (resp. an affine $C^r\mathfrak{S}$ manifold) and that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ are $C^r\mathfrak{S}$ maps.
(2) Let G be a $C^r\mathfrak{S}$ group. A subgroup K of G is said to be a $C^r\mathfrak{S}$ *subgroup of* G if K is a $C^r\mathfrak{S}$ submanifold of G.
(3) Let G and G' be $C^r\mathfrak{S}$ groups. A group homomorphism $G \longrightarrow G'$ is called a $C^r\mathfrak{S}$ *group homomorphism* if it is a $C^r\mathfrak{S}$ map. A $C^r\mathfrak{S}$ group homomorphism $f : G \longrightarrow G'$ is said to be a $C^r\mathfrak{S}$ *group isomorphism* if there exists a $C^r\mathfrak{S}$ group homomorphism $h : G' \longrightarrow G$ such that $f \circ h = id$ and $h \circ f = id$.

(4) A *representation* of a $C^r\mathfrak{S}$ group G means a group homomorphism from G to some $GL(\mathbb{R}^n)$ which is of class $C^r\mathfrak{S}$. We use a representation as a representation space.

Remark that a $C^\omega\mathfrak{S}$ group is called a *Nash group* [10] (resp. an *exponentially Nash group* [8]) if M is $\mathfrak{R}$ (resp. $\mathbf{R}_{\exp}$). Moreover remark that one-dimensional connected Nash groups are classified by J. J. Madden and C. H. Stanton [10].

DEFINITION 2.5. Let r be a non-negative integer, ∞ or ω and let G be a $C^r\mathfrak{S}$ group.
(1) A $C^r\mathfrak{S}$ submanifold in a representation Ω of G is called a *$C^r\mathfrak{S}$-G submanifold of* Ω if it is G invariant.
(2) A *$C^r\mathfrak{S}$-G manifold* is a pair (X, θ) consisting of a $C^r\mathfrak{S}$ manifold and a group action θ of G on X such that $\theta : G \times X \longrightarrow X$ is a $C^r\mathfrak{S}$ map. For simplicity of notation, we write X instead of (X, θ).
(3) Let X and Y be $C^r\mathfrak{S}$-G manifolds. A $C^r\mathfrak{S}$ map $f : X \longrightarrow Y$ is called a *$C^r\mathfrak{S}$-G map* if it is a G map. A $C^r\mathfrak{S}$-G map $h : X \longrightarrow Y$ is said to be a *$C^r\mathfrak{S}$-G diffeomorphism* (an *$\mathfrak{S}$-G homeomorphism* if $r = 0$) if there exists a $C^r\mathfrak{S}$-G map $k : Y \longrightarrow X$ such that $h \circ k = id$ and $k \circ h = id$.
(4) We say that a $C^r\mathfrak{S}$-G manifold is an *affine $C^r\mathfrak{S}$-G manifold* if it is $C^r\mathfrak{S}$-G diffeomorphic ($\mathfrak{S}$-G homeomorphic if $r = 0$) to a $C^r\mathfrak{S}$-G submanifold of some representation of G.

Let G be an affine $C^\infty\mathfrak{S}$ group. Then notice that every closed subgroup of G which is an $\mathfrak{S}$ set is a $C^\infty\mathfrak{S}$ subgroup by the fact that each closed subgroup of a Lie group is a Lie subgroup of it.

Note that for any M, a Nash group (resp. an affine Nash group) is a $C^\omega\mathfrak{S}$ group (resp. an affine $C^\omega\mathfrak{S}$ group), and that a Nash G manifold (resp. an affine Nash G manifold) is a $C^\omega\mathfrak{S}$-G manifold (resp. an affine $C^\omega\mathfrak{S}$-G manifold).

THEOREM 2.6 (Theorem 1 page 54 [17]). *Let $U \subset \mathbb{R}^n$ be an open set and let $V \subset \mathbb{R}^n$ be a nonsingular algebraic set. If $f : U \longrightarrow \mathbb{R}$ is a C^∞ function such that $f|U \cap V$ is a polynomial function, then for each compact set $H \subset U, \epsilon > 0, q \in \mathbb{N}$, there exists a polynomial function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that:*

(a) $\max_{x \in H} |\frac{\partial^r (F-f)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(x)| < \epsilon,\ r = i_1 + \cdots + i_n,\ 0 \le r \le q.$

(b) $F|V \cap U = f|V \cap U$.

We use the following remark to prove a slice theorem in the equivariant o-minimal category.

REMARK 2.7. Let r be ω or ∞. Then every compact affine $C^r\mathfrak{S}$ group G is $C^r\mathfrak{S}$-G diffeomorphic to a $C^r\mathfrak{S}$-G submanifold of some representation of G.

Proof. We only have to prove the result when $r = \omega$ because the other case is proved similarly. Since G is compact, G can be $C^\infty G$ imbeddable into some representation Ω of G. Let i denote a $C^\infty G$ imbedding $G \longrightarrow \Omega$. We may assume that G is a $C^\omega\mathfrak{S}$ submanifold of some $\mathbb{R}^n$ because G is affine. By Theorem 2.6, there exists a polynomial map $F : \mathbb{R}^n \longrightarrow \Omega$ such that $F|G$ is an approximation of i. Since G is compact, averaging F, we may assume that $F|G$ is a polynomial G map. If this approximation is sufficiently close, then $F|G$ is a $C^\infty G$ imbedding by 1.4 [4], hence $F|G$ is a $C^\omega\mathfrak{S}$-G imbedding. Thus G is $C^\omega\mathfrak{S}$-G diffeomorphic to a $C^\omega\mathfrak{S}$-G submanifold $F(G)$ of Ω. □

Recall universal G vector bundles (cf. [7]).

DEFINITION 2.8. Suppose that G is a compact Lie group. Let Ω be an n-dimensional representation of G and B the representation map $G \longrightarrow GL_n(\mathbb{R})$ of Ω. Suppose further that $M(\Omega)$ denotes the vector space of (n,n)-matrices with the action $(g, A) \in G \times M(\Omega) \longrightarrow B(g)^{-1}AB(g) \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\},$$
$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\},$$

$$u : E(\Omega, k) \longrightarrow G(\Omega, k) : u((A, v)) = A,$$

where A' denotes the transposed matrix of A. Then $G(\Omega, k)$ and $E(\Omega, k)$ are algebraic sets. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and k*. Since $G(\Omega, k)$ and $E(\Omega, k)$ are nonsingular, $\gamma(\Omega, k)$ is a Nash G vector bundle.

The following proposition is obtained in a similar way of the usual equivariant Nash cases [9].

PROPOSITION 2.9. *Let G be a compact affine $C^\omega\mathfrak{S}$ group and let X be a $C^\omega\mathfrak{S}$-G submanifold of a representation Ω of G. Then there exists a $C^\omega\mathfrak{S}$-G tubular neighborhood (U, p) of X in Ω.*

Notice that Proposition 2.9 remains valid in the $C^\infty\mathfrak{S}$ category.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we recall the following two results [3].

THEOREM 3.1 [3]. *Let $A \subset \mathbb{R}^n$ be a closed $\mathfrak{S}$ set and let r be a non-negative integer. Then there exists a $C^r\mathfrak{S}$ function f on $\mathbb{R}^n$ with $A = f^{-1}(0)$.*

COROLLARY 3.2. *Let X be an affine $C^r\mathfrak{S}$ $(0 \leq r < \infty)$ manifold. Then X can be $C^r\mathfrak{S}$ imbedded into some $\mathbb{R}^n$ such that X is closed in $\mathbb{R}^n$. Moreover it is possible to $C^r\mathfrak{S}$ imbed into some $\mathbb{R}^k$ such that X is bounded and $\overline{X} - X$ consists of at most one point.*

Proof. We may assume that X is noncompact because the result is clear when X is compact. Let X be a $C^r\mathfrak{S}$ submanifold of $\mathbb{R}^{n-1}$. By Proposition 2.3, $\overline{X} - X$ is a closed $\mathfrak{S}$ set. Applying Theorem 3.1, we have a $C^r\mathfrak{S}$ function f on $\mathbb{R}^{n-1}$ with $\overline{X} - X = f^{-1}(0)$. Hence considering the graph of $1/f$ on X in place of X, we obtain the first half of the corollary. For the latter half, suppose that X is contained and closed in $\mathbb{R}^n$. Let $s : \mathbb{R}^n \longrightarrow S^n \subset \mathbb{R}^{n+1}$ be the stereographic projection. Then $s(X)$ satisfies the requirements of the latter half. □

Recall that a subset of $\mathbb{R}^n$ is *locally closed* if it is the intersection of an open set and a closed set.

PROPOSITION 3.3 [3]. *Suppose that $X \subset \mathbb{R}^n$ is a locally closed $\mathfrak{S}$ set and that f and g are $C^0\mathfrak{S}$ functions on X with $f^{-1}(0) \subset g^{-1}(0)$. If*

M is polynomially bounded, then there exist an integer N and a $C^0\mathfrak{S}$ function $h : X \longrightarrow \mathbb{R}$ such that $g^N = hf$ on X. In particular, for any compact subset K of X, there exists a positive constant c such that $|g^N| \leq c|f|$ on K

Proof of Theorem 1.1. Let X be a $C^r\mathfrak{S}$ manifold. If dim $X = 0$ then X consists of finitely many points. Thus the result holds.

Assume that $m :=$ dim $X \geq 1$. Let $\{\phi_i : U_i \longrightarrow \mathbb{R}^m\}_{i=1}^l$ be a $C^r\mathfrak{S}$ atlas of X. Then each $\phi_i(U_i)$ is a noncompact $C^r\mathfrak{S}$ submanifold of $\mathbb{R}^m$. Hence by Corollary 3.2, we have a $C^r\mathfrak{S}$ imbedding $\phi_i' : \phi_i(U_i) \longrightarrow \mathbb{R}^{m'}$ such that the image is bounded in $\mathbb{R}^{m'}$ and

$$\overline{\phi_i' \circ \phi_i(U_i)} - \phi_i' \circ \phi_i(U_i)$$

consists of one point, say 0. Set

$$\eta : \mathbb{R}^{m'} \longrightarrow \mathbb{R}^{m'}, \eta(x_1, \cdots, x_{m'}) = \left(\sum_{j=1}^{m'} x_j^{2k} x_1, \cdots, \sum_{j=1}^{m'} x_j^{2k} x_{m'}\right),$$

$$g_i : U_i \longrightarrow \mathbb{R}^{m'}, \eta \circ \phi_i' \circ \phi_i,$$

for a sufficiently large integer k. Then g_i is a $C^r\mathfrak{S}$ imbedding of U_i into $\mathbb{R}^{m'}$. Moreover the extension $\tilde{g}_i : X \longrightarrow \mathbb{R}$ of g_i is defined by $\tilde{g}_i = 0$ on $X - U_i$.

We now prove that $\tilde{g}_i$ is of class $C^r\mathfrak{S}$. It is sufficient to see this on each $C^r\mathfrak{S}$ coordinate neighborhood of X. Hence we may assume that X is open in $\mathbb{R}^m$. We only have to prove that for any sequence $\{a_j\}_{j=1}^\infty$ in U_i convergent to a point of $X - U_i$ and for any $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq r$, $\{D^\alpha g_i(a_j)\}_{j=1}^\infty$ converges to 0. On the other hand, $g_i = (\sum_{j=1}^{m'} \phi_{ij}^{2k}\phi_{i1}, \cdots \sum_{j=1}^{m'} \phi_{ij}^{2k}\phi_{im'})$, where $\phi_i' \circ \phi_i = (\phi_{i1}, \cdots, \phi_{im'})$. Each ϕ_{ij} is bounded, and every $\{\phi_{ij}(a_i)\}_{i=1}^\infty$ converges to 0, and

$$\begin{aligned}
&|D^\alpha(\phi_{ij}^{2k}\phi_{is})| \\
&= |\sum_{\beta+\gamma=\alpha} (\alpha!/(\beta!\gamma!))D^\beta\phi_{ij}^{2k}D^\gamma\phi_{is}| \\
&\leq C \sum_{\beta_1+\cdots+\beta_{l'}+\gamma=\alpha, \beta_i\neq 0} |\phi_{ij}^{2k-l'}D^{\beta_1}\phi_{ij}\cdots D^{\beta_{l'}}\phi_{ij}D^\gamma\phi_{is}| \\
&\leq C'|\phi_{ij}^{2k-\gamma}|\psi,
\end{aligned}$$

where C, C' are constants, and ψ is the positive $C^0\mathfrak{S}$ function defined by

$$\psi(x) = \max\left\{1, \sum_{\beta_1+\cdots+\beta_{l'}+\gamma=\alpha} |D^{\beta_1}\phi_{ij}(x)\cdots D^{\beta_{l'}}\phi_{ij}(x)D^{\gamma}\phi_{is}(x)|\right\}.$$

Define

$$\theta_{ij}(x) = \begin{cases} \min\{|\phi_{ij}(x)|, 1/\psi(x)\} & \text{on } U_i \\ 0 & \text{on } X - U_i, \end{cases}$$

$$\tilde{\phi_{ij}} = \begin{cases} \phi_{ij} & \text{on } U_i \\ 0 & \text{on } X - U_i. \end{cases}$$

Then θ_{ij} and $\tilde{\phi_{ij}}$ are $C^0\mathfrak{S}$ functions on X such that

$$X - U_i \subset \theta_{ij}^{-1}(0) = \tilde{\phi_{ij}}^{-1}(0).$$

Hence by Proposition 3.3 we have $|\tilde{\phi_{ij}}^{l''}| \le d\theta_{ij}$ on some open $\mathfrak{S}$ neighborhood V of $X - U_i$ in X for some integer l'', where d is a constant.

On the other hand, by the definition of θ_{ij},

$$|\psi\theta_{ij}| \le 1 \text{ on } U_i.$$

Hence the above argument proves that

$$|D^{\alpha}(\phi_{ij}^{2k}\phi_{is})| \le c'|\phi_{ij}^{2k-r-l''}|$$

on $U_i \cap V$, where c' is a constant and we take k such that $2k \ge r + l'' + 1$. Hence each $\tilde{g}_i$ is of class $C^r\mathfrak{S}$. It is easy to see that

$$\prod_{i=1}^{l} \tilde{g}_i : X \longrightarrow \mathbb{R}^{lm'}$$

is a $C^r\mathfrak{S}$ imbedding. □

4. Proof of Theorem 1.2

For the proof of Theorem 1.2, we prove the following slice theorem in the equivariant o-minimal category. Equivariant C^∞ version of it is well known (cf. [5])

THEOREM 4.1. *Suppose that G is a compact affine $C^\infty\mathfrak{S}$ group, X is a $C^\infty\mathfrak{S}$-G manifold, and that $x \in X$. Then there exists a linear $C^\infty\mathfrak{S}$ slice at x in X.*

To prove Theorem 4.1, we need several definitions and results. Our proof of Theorem 4.1 is a modification of that of Theorem 2.5 [6].

Let $A \subset \mathbb{R}^m$ be an $\mathfrak{S}$ set. A $C^0\mathfrak{S}$ map $f : A \longrightarrow \mathbb{R}^n$ is called $\mathfrak{S}$ *trivial* if there exist $k \in \mathbb{N}$ and a $C^0\mathfrak{S}$ map $g : A \longrightarrow \mathbb{R}^k$ such that $a \mapsto (f(a), g(a))$ is a homeomorphism of A onto $f(A) \times g(A)$.

THEOREM 4.2 (Local triviality) [3]. *Let $A \subset \mathbb{R}^m$ be an $\mathfrak{S}$ set and let $f : A \longrightarrow \mathbb{R}^n$ be a $C^0\mathfrak{S}$ map. Then there exists a partition $\{C_1, \cdots, C_l\}$ of $f(A)$ into $\mathfrak{S}$ sets such that each $f|f^{-1}(C_i)$ is $\mathfrak{S}$ trivial.*

Let G be a compact Lie group and let Ω be a representation of G. Then the algebra $\mathbb{R}[\Omega]^G$ of G invariant polynomials on Ω is finitely generated [18]. Let $p_1, \cdots, p_l$ be G invariant polynomials on Ω which generate $\mathbb{R}[\Omega]^G$ and let

$$p : \Omega \longrightarrow \mathbb{R}^l, p(x) = (p_1(x), \cdots, p_l(x)).$$

Then $p : \Omega \longrightarrow \mathbb{R}^l$ is a proper polynomial map. Notice that every polynomial map is an $\mathfrak{S}$ map in any M. Suppose that X is a G invariant $\mathfrak{S}$ subset of Ω. Then $p(\Omega)$ and $p(X)$ are $\mathfrak{S}$ subsets of $\mathbb{R}^l$. Moreover p induces the map $j : \Omega/G \longrightarrow \mathbb{R}^l$ such that $p = j \circ \pi$, where $\pi : \Omega \longrightarrow \Omega/G$ denotes the orbit map. Then j is a closed imbedding. Hence we identify X/G ($\subset \Omega/G$) with an $\mathfrak{S}$ subset $j(X/G)$ ($\subset j(\Omega/G)$) of $\mathbb{R}^l$.

PROPOSITION 4.3. *Let G be a compact affine $C^\infty\mathfrak{S}$ group and let X be a G invariant $\mathfrak{S}$ subset of a representation of G. Then the orbit space X/G admits a structure of $\mathfrak{S}$ set such that:*

(a) *The orbit map $\pi : X \longrightarrow X/G$ is an $\mathfrak{S}$ map and there exists a decomposition of X/G into finitely many $\mathfrak{S}$ sets $T_1, \cdots, T_s$ such that each $\pi|\pi^{-1}(T_k) : \pi|\pi^{-1}(T_k) \longrightarrow T_k$ admits a $C^0\mathfrak{S}$ section.*

(b) *For any map f from X/G to any $\mathfrak{S}$ set Y, f is an $\mathfrak{S}$ map if and only if so is $f \circ \pi$.*

Proof. By the construction of definable structure of X/G, π is an $\mathfrak{S}$ map. By Theorem 4.2, there exists a decomposition of X/G into finitely many $\mathfrak{S}$ sets $\{T_k\}$ such that each $\pi|\pi^{-1}(T_k) : \pi^{-1}(T_k) \longrightarrow T_k$ admits a $C^0\mathfrak{S}$ section s_k. Hence $f|T_k = f \circ (\pi \circ s_k) = (f \circ \pi) \circ s_k$. Thus f is an $\mathfrak{S}$ map if so is $f \circ \pi$, this proves Property (b). □

PROPOSITION 4.4. *Let G be a compact affine $C^\infty\mathfrak{S}$ group. Suppose that X is an affine $C^\infty\mathfrak{S}$-G manifold with free action. Then the orbit space X/G admits an affine $C^\infty\mathfrak{S}$ manifold structure such that:*

(a) *The orbit map $\pi : X \longrightarrow X/G$ is a $C^\infty\mathfrak{S}$ map.*
(b) *For any map f from X/G to any affine $C^\infty\mathfrak{S}$ manifold Y, f is a $C^\infty\mathfrak{S}$ map.*

if and only if so is $f \circ \pi$.

Proof. Since G is compact and by a fundamental fact of $C^\infty G$ manifolds, X/G is a C^∞ manifold, π is a C^∞ map, for any map $f : X/G \longrightarrow Y$, f is of class C^∞ if and only if so is $f \circ \pi$. Notice that X/G is an $\mathfrak{S}$ set. Hence X/G is an affine $C^\infty\mathfrak{S}$ manifold. By construction, π is an $\mathfrak{S}$ map. Hence π is a $C^\infty\mathfrak{S}$ map. Applying Theorem 4.2 to π, f is an $\mathfrak{S}$ map if so is $f \circ \pi$, hence f is of class $C^\infty\mathfrak{S}$ if and only if so is $f \circ \pi$. Therefore our proposition is proved. □

PROPOSITION 4.5. *Let G be a compact affine $C^\infty\mathfrak{S}$ group. Suppose that K is a compact $C^\infty\mathfrak{S}$ subgroup of G and that S is an affine $C^\infty\mathfrak{S}$-K manifold. Then the twisted product $G \times_K S$ with the standard G action $G \times (G \times_K S) \longrightarrow G \times_K S, (g, [g', s]) \mapsto [gg', s]$ is a $C^\infty\mathfrak{S}$-G manifold.*

Proof. The product K manifold $G \times S$ with the K action $(k, (g, s)) \mapsto (gk^{-1}, ks)$ is an affine $C^\infty\mathfrak{S}$-K manifold by Remark 2.7. Since the action of K on $G \times S$ is free and by Proposition 4.4, $G \times_K S$ is an affine $C^\infty\mathfrak{S}$ manifold. Clearly the standard action of G on $G \times_K S$ is of class C^∞. Let $\pi : G \times S \longrightarrow G \times_K S$ be the orbit map. Then π is a $C^\infty\mathfrak{S}$ map, in particular it is an $\mathfrak{S}$ map. Clearly the action of G on $G \times S$

defined by $G\times(G\times S)\longrightarrow G\times S, (g,(g',s))\mapsto (gg',s)$ is of class $C^\infty\mathfrak{S}$. Since the graph of the standard action on $G\times_K S$ is the image of that of the action map of G on $G\times S$ by $id_G\times\pi\times\pi$, it is an $\mathfrak{S}$ set, thus it is of class $C^\infty\mathfrak{S}$. □

We define $C^\infty\mathfrak{S}$ slices and linear $C^\infty\mathfrak{S}$ slices in a similar way of smooth slices and linear smooth slices (cf. [5]).

DEFINITION 4.6. Let G be a compact affine $C^\infty\mathfrak{S}$ group. Suppose that X is a $C^\infty\mathfrak{S}$-G manifold, and that K is a compact $C^\infty\mathfrak{S}$ subgroup of G.

(1) We say that a K invariant $C^\infty\mathfrak{S}$ submanifold S of X is a $C^\infty\mathfrak{S}$-K *slice* if GS is open in X, S is affine as a $C^\infty\mathfrak{S}$-K manifold, and

$$\mu : G\times_K S\longrightarrow GS\ (\subset X), [g,x]\mapsto gx$$

is a $C^\infty\mathfrak{S}$-G diffeomorphism.

Remark that μ is always an $\mathfrak{S}$ map because its graph is the image of that of $G\times S\longrightarrow GS, (g,s)\mapsto gs$ by $\pi\times id_{GS}$, where π denotes the orbit map $G\times S\longrightarrow G\times_K S$.

(2) A $C^\infty\mathfrak{S}$-K slice S is called *linear* if there exist a representation Ω of K and a $C^\infty\mathfrak{S}$-K imbedding $j:\Omega\longrightarrow X$ such that $j(\Omega)=S$.

(3) We say that a $C^\infty\mathfrak{S}$-K slice (resp. a linear $C^\infty\mathfrak{S}$-K slice) S is a $C^\infty\mathfrak{S}$ *slice* (resp. a *linear* $C^\infty\mathfrak{S}$ *slice*) at x in X if $K=G_x$ and $x\in S$ (resp. $K=G_x$ and $j(0)=x$).

PROPOSITION 4.7. *Let G be a compact affine $C^\infty\mathfrak{S}$ group and let K be a compact $C^\infty\mathfrak{S}$ subgroup of G. Suppose that $\pi : G\longrightarrow G/K$ denotes the orbit map. Then there exist an open K invariant $\mathfrak{S}$ neighborhood V of eK in G/K and a $C^\infty\mathfrak{S}$-K section $\sigma : V\longrightarrow G$ such that $\sigma(eK)=e$ and that V is $C^\infty\mathfrak{S}$-K diffeomorphic to some representation of K. Here the actions of K on G/K and G are the following:*

$$K\times G/K\longrightarrow G/K, (k,gK)\mapsto kgK,$$

$$K\times G\longrightarrow G, (k,g)\mapsto kgk^{-1}.$$

Proof. At first we show that the left coset space G/K becomes a $C^\infty\mathfrak{S}$-K manifold. By Proposition 4.4, G/K becomes an affine $C^\infty\mathfrak{S}$ manifold such that the orbit map $\pi : G \longrightarrow G/K$ is a $C^\infty\mathfrak{S}$ map. Since the action map of K on G/K is the image of that of K on G defined by $K \times G \longrightarrow G, (k,g) \mapsto kg$, the action map of K on G/K is of class $\mathfrak{S}$. Thus G/K is a $C^\infty\mathfrak{S}$-K manifold.

Furthermore, $\pi : G \longrightarrow G/K$ is a $C^\infty\mathfrak{S}$-K map under the above K actions, and $e \in G$ and $eK \in G/K$ are fixed points of K. Thus the tangent spaces $T_e(G)$ at $e \in G$ and $T_{eK}(G/K)$ at $eK \in G/K$ are representations of K. Since K is compact, we may assume that they are orthogonal representation spaces. Since π is submersive, the differential $(d\pi)_e : T_e(G) \longrightarrow T_{eK}(G/K)$ is a surjective linear K map.

The tangent space $T_e(K)$ of K at $e \in K$ is a K invariant linear subspace of $T_e(G)$. Let L denote the orthogonal complement to $T_e(K)$ in $T_e(G)$. Then L is a K invariant linear subspace of $T_e(G)$ and $T_e(G) = T_e(K) \oplus L$. Moreover $q := \dim L = \dim G - \dim K = \dim G/K$, and $(d\pi)_e|L : L \longrightarrow T_e(G/K)$ is a linear K isomorphism.

Using the exponential map, one can find a q-dimensional K invariant C^∞ submanifold V^* of G such that $e \in V^*$ and $T_e(V^*) = L$.

Approximating V^*, we now construct q-dimensional K invariant $C^\infty\mathfrak{S}$ submanifold $V' \ni e$ of G such that $T_e(G) = T_e(K) \oplus T_e(V')$ and $(d\pi)_e|T_e(V') : T_e(V') \longrightarrow T_e(G/K)$ is a linear K isomorphism. Recall that G can be regarded as a $C^\infty\mathfrak{S}$-G submanifold of some representation Ω of G by Remark 2.7. Take a $C^\infty K$ tubular neighborhood (W,p) of V^* in G. Let $\phi : W \longrightarrow G(\Omega, \dim K), \phi(x) =$ the normal space of V^* in G at $p(x)$ and $\psi : W \longrightarrow \Omega, \psi(x) = x - p(x)$. Define

$$\Phi : W \longrightarrow \gamma(\Omega, \dim K)\ (\subset G(\Omega, \dim K) \times \Omega \subset M(\Omega) \times \Omega),$$
$$\Phi(x) = (\phi(x), \psi(x)).$$

Then $V^* = \Phi^{-1}(G(\Omega, \dim K))$, and Φ is a $C^\infty K$ map and transverse to $G(\Omega, \dim K\ )$. By Theorem 2.6, there exists a polynomial map $h : W' \longrightarrow M(\Omega) \times \Omega$ such that h approximates Φ and $h(e) = \Phi(e)$, where W' is an appropriate compact $C^\infty\mathfrak{S}$-K manifold with $W' \ni e$ and $W' \subset W$. By averaging h, we may assume that h is a polynomial

K map. Using Proposition 2.9, one can find a $C^\omega\mathfrak{S}$-K tubular neighborhood $(T,\overline{p})$ of $\gamma(\Omega, \dim K)$ in $M(\Omega)\times\Omega$. If this approximation is sufficiently close, the image of h lies in T. Hence composing h with $\overline{p}$, we have a $C^\infty\mathfrak{S}$-G map $\overline{h} : W' \longrightarrow \gamma(\Omega, \dim K)$ with $\overline{h}(e) = \Phi(e)$ as an approximation of $\Phi|W'$. Thus $\overline{h}$ is transverse to $G(\Omega, \dim K)$. Hence $V' := (\overline{h})^{-1}(G(\Omega, \dim K)) \cap \operatorname{Int} W'$ is a $C^\infty\mathfrak{S}$-K submanifold of G contained in W. If this approximation is sufficiently close, then V' has the required properties.

Hence $\pi|V' : V' \longrightarrow G/K$ is a $C^\infty\mathfrak{S}$-K map and the differential $(d(\pi|V'))_e : T_e(V') \longrightarrow T_e(G/K)$ is a linear K isomorphism. By the inverse function theorem, there exists an open $\mathfrak{S}$ neighborhood U of e in V' such that $\pi|U : U \longrightarrow \pi(U)$ is a $C^\infty\mathfrak{S}$ diffeomorphism onto an open $\mathfrak{S}$ neighborhood $V := \pi(U)$ of eK in G/K. Since G is affine and K is compact, shrinking U, if necessary, we may assume that U is $C^\infty\mathfrak{S}$-K diffeomorphic to some representation of K. Therefore

$$\sigma := (\pi|U)^{-1} : V \longrightarrow U \subset G$$

is a $C^\infty\mathfrak{S}$-K section over V of π and $\sigma(eK) = e$. □

We prepare the following two lemmas obtained by a way similar to the proof of 1.2 [6] and 5.2 [5], respectively.

Lemma 4.8. *Let G be an affine $C^\infty\mathfrak{S}$ group and let H be a compact $C^\infty\mathfrak{S}$ subgroup of G. Suppose that U is an open neighborhood of H in G. Then the identity element $e \in G$ has an open $\mathfrak{S}$ neighborhood $V \subset U$ such that $HVH = V$.*

Lemma 4.9. *Let G be a compact affine $C^\infty\mathfrak{S}$ group and let K be a compact $C^\infty\mathfrak{S}$ subgroup of G. Suppose that S is a K invariant $C^\infty\mathfrak{S}$ submanifold of a $C^\infty\mathfrak{S}$-G manifold of X such that S is affine as a $C^\infty\mathfrak{S}$-K manifold. Then the following are equivalent.*

(1) *S is a $C^\infty\mathfrak{S}$-K slice in X.*
(2) *GS is open in X and there exists a $C^\infty G$ map $\gamma : GS \longrightarrow G/K$ such that $\gamma^{-1}(eK) = S$.*

Proof of Theorem 4.1. By the definition of $C^\infty\mathfrak{S}$ manifolds, there exists a $C^\infty\mathfrak{S}$ diffeomorphism f from an open $\mathfrak{S}$ neighborhood V of x in X to an open $\mathfrak{S}$ neighborhood B of the origin of $\mathbb{R}^m$, where $m = \dim X$. Let n be the dimension of a $C^\infty\mathfrak{S}$ submanifold $f(G(x)\cap V)$ of $\mathbb{R}^m$. Then the set N' of points in $\mathbb{R}^m$ whose inner product with every point in the tangent space $T_0 f(G(x)\cap V))$ is zero is a subspace of dimension $m-n$. Hence $N := B\cap N'$ is a $C^\infty\mathfrak{S}$ submanifold of $\mathbb{R}^m$. Thus $S^* := f^{-1}(N)$ is a $C^\infty\mathfrak{S}$ submanifold of V. Moreover we have $T_x X = T_x G(x) \oplus T_x(f^{-1}(N))$ because $T_0\mathbb{R}^m = T_0 f(G(x)\cap V)\oplus T_0 N$ and f is a $C^\infty\mathfrak{S}$ diffeomorphism. On the other hand, X admits a G_x invariant C^∞ metric because G_x is compact. Hence using this metric, S^* is G_x invariant by the proof of 2.3 [6].

By Proposition 4.7, one can find an open G_x invariant $\mathfrak{S}$ neighborhood W of eG_x in G/G_x and a $C^\infty\mathfrak{S}$-G_x section $\gamma : W \longrightarrow G$ of the orbit map $\pi : G \longrightarrow G/G_x$ such that $\gamma(eG_x) = e$. Define

$$F : W\times S^* \longrightarrow X, F(w,s) = \gamma(w)s.$$

Since $F = (\phi|G\times S^*)\circ(\gamma\times id_{S^*})$, F is a $C^\infty\mathfrak{S}$ map, where ϕ denotes the group action map $G\times X \longrightarrow X$. The map $\alpha_x : G/G_x \longrightarrow G(x)$ defined by $\alpha(gG_x) = gx$ is a C^∞ diffeomorphism, and its graph is an $\mathfrak{S}$ set because it is the image of the graph of the map $G \longrightarrow G(x), g\mapsto gx$ by $\pi\times id_{G(x)}$. Therefore α_x is a $C^\infty\mathfrak{S}$ diffeomorphism.

We identify $T_{(eG_x,x)}(W\times S^*)$ with $T_{eG_x}(W)\oplus T_x S^*$. Then $dF_{(eG_x,x)}(y_1,y_2) = (d\alpha_x)_{eG_x}(y_1) + (did)_x(y_2), (y_1,y_2)\in T_{eG_x}W\oplus T_x S^*$. Since $T_x X = T_x G(x)\oplus T_x S^*$, and since $(d\alpha_x)_{eG_x}$ and $(did)_x$ are isomorphisms, $dF_{(eG_x,x)}$ is an isomorphism. By the inverse function theorem, there exist open $\mathfrak{S}$ neighborhoods $U'\subset W$ of eG_x and $W'\subset S^*$ of x such that $F|(U'\times W')$ is a $C^\infty\mathfrak{S}$ diffeomorphism onto an open $\mathfrak{S}$ neighborhood of x in X. Since G_x is compact, x has an open G_x invariant neighborhood $S'\subset W'$ ($\subset S^*$).

Let $U_0 = \pi^{-1}(U')$. Then U_0 is an open $\mathfrak{S}$ neighborhood of G_x in G. Since G_x is compact and by Lemma 4.8, there exists an open $\mathfrak{S}$ neighborhood $W'\subset U_0$ of e such that $G_x W' G_x = W'$. For any subsets $A, B\subset X$, let $G(A|B)$ denote $\{g\in G | gB\cap A\neq\emptyset\}$. Since W' is an open $\mathfrak{S}$ neighborhood of G_x and by 1.1.6 [14], x has an open neighborhood V_0 with $G(V_0|V_0)\subset W'$. Shrinking V_0, if necessary, we may assume that V_0 is an $\mathfrak{S}$ set. Thus $G(G_xV_0|G_xV_0) = G_x(G(V_0|V_0))G_x \subset U_0$. Hence $S := S' \cap G_xV_0$ is an open G_x invariant $C^\infty\mathfrak{S}$ submanifold of S^* containing x. Moreover since G_x is compact, shrinking S, if necessary, we can find a $C^\infty\mathfrak{S}$-G_x imbedding j from some representation Ω' of G_x into X such that $j(\Omega') = S$ and $j(0) = x$. Since $F|U' \times W' : U' \times W' \longrightarrow F(U' \times W')$ is a homeomorphism, $U_0S = F(U' \times S)$ is open in X. Thus $GS = GU_0S$ is open in X. Moreover the map $l : GS \longrightarrow G/G_x$ defined by $gs \mapsto gG_x$ is well-defined and $S = l^{-1}(eG_x)$. Clearly l is a G map. By the proof of 2.5 [6], l is of class C^∞.

Therefore by Lemma 4.9, S is the required linear $C^\infty\mathfrak{S}$ slice at x.□

To prove Theorem 1.2, we need the following two lemmas.

LEMMA 4.10. *Let G be a compact affine $C^\infty\mathfrak{S}$ group. Suppose that X is a $C^\infty\mathfrak{S}$-G manifold and $x \in X$. Then there exists a linear $C^\infty\mathfrak{S}$ slice S at x in X such that $G \times_{G_x} S$ is affine.*

Proof. By Theorem 4.1, there exists a linear $C^\infty\mathfrak{S}$ slice S' at x in X. Let Ξ be a representation of G_x and let $G \times_{G_x} \Xi \longrightarrow GS' \subset X$ be a $C^\infty\mathfrak{S}$-G diffeomorphism. Since G is compact and by a fundamental fact, $G \times_{G_x} \overline{D_2}$ can be $C^\infty G$ imbeddable into some representation Ω of G, where $\overline{D_2}$ denotes $\{x \in \Xi \,|\, ||x|| \leq 2\}$. Let i denote a $C^\infty G$ imbedding $G \times_{G_x} \overline{D_2} \longrightarrow \Omega$. On the other hand, we may assume that $G \times_{G_x} \mathrm{Int}\, \overline{D_2}$ is a $C^\infty\mathfrak{S}$ submanifold of some $\mathbb{R}^n$ by Proposition 4.4. By Theorem 2.6, there exists a polynomial map $F : \mathbb{R}^n \longrightarrow \Omega$ such that $F|(G \times_{G_x} D)$ is an approximation of i, where D denotes the open unit ball of Ξ. Since G is compact, averaging F, we may assume that $F|(G \times_{G_x} D)$ is a polynomial G map. If this approximation is sufficiently close, then $F|(G \times_{G_x} D)$ is a $C^\infty G$ imbedding by 1.4 [4]. Thus $G \times_{G_x} D$ is $C^\infty\mathfrak{S}$-G diffeomorphic to a $C^\infty\mathfrak{S}$-G submanifold $F(G \times_{G_x} D)$ of Ω. Since D is $C^\infty\mathfrak{S}$-G diffeomorphic to Ξ, we complete the proof. □

LEMMA 4.11. *Let M be exponential and let G be a compact affine $C^\infty\mathfrak{S}$ group. Suppose that D_1 and D_2 are open balls in a representation Ω of G of radius a and b with same center the origin and $a < b$. If $A, B \in \mathbb{R}, A \neq B$, then there exists a G invariant $C^\infty\mathfrak{S}$ function f on Ω such that $f = A$ on D_1 and $f = B$ on $\Omega - \overline{D_2}$.*

Proof. We may assume that $A = 1$ and $B = 0$. We now construct the desired function when $\Omega = \mathbb{R}$. We may suppose that $D_1 = (-a, a)$ and $D_2 = (-b, b)$ are open intervals. Recall the $C^\infty\mathfrak{S}$ function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

(See Example 2.2 (1)). The function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(x) = F(b - x)F(b + x)/(F(b - x)F(b + x) + F(x^2 - a^2))$$

is the desired function.

Hence for a general representation Ω of G, $f : \Omega \longrightarrow \mathbb{R}$, $f(x) = \psi(|x|)$ is the required function, where $|x|$ denotes the standard norm of x in Ω. □

Proof of Theorem 1.2. Let X be a compact $C^\infty\mathfrak{S}$-G manifold and let $x \in X$. By Theorem 4.1, there exists a linear slice $G \times_{G_x} \Omega_x$ at x in X. Furthermore we may assume that $G \times_{G_x} \Omega_x$ is $C^\infty\mathfrak{S}$-G diffeomorphic to a $C^\infty\mathfrak{S}$-G submanifold of some representation Ξ_x of G by Lemma 4.10. Let $\phi_x : G \times_{G_x} \Omega_x \ (\subset X) \longrightarrow \Xi_x$ be a $C^\infty\mathfrak{S}$-G imbedding.

Since M is exponential and by Lemma 4.11, there exists a G invariant $C^\infty\mathfrak{S}$ function $f_x : \Omega_x \longrightarrow \mathbb{R}$ such that $f_x|D_1 = 1$ and $f_x|(\Omega - \overline{D_2}) = 0$, where D_r denotes the open ball of radius r with center the origin. Using f_x, we can extend $\phi_x|G \times_{G_x} D_1 : G \times_{G_x} D_1 \longrightarrow \Xi_x$ to a $C^\infty\mathfrak{S}$-G map $\psi_x : X \longrightarrow \Xi_x$, and we can find a G invariant $C^\infty\mathfrak{S}$ map $h_x : X \longrightarrow \mathbb{R}$ such that $h_x|(G \times_{G_x} D_1) = 1$ and $h_x|(X - G \times_{G_x} \overline{D_2}) = 0$. Since X is compact, one can find $x_1, \cdots, x_l \in X$ such that $X = \cup_{i=1}^{l}(G \times_{G_{x_i}} D_1)$. Therefore $\psi : X \longrightarrow \Xi_{x_1} \times \cdots \times \Xi_{x_l} \times \mathbb{R}^l, \psi(x) = (\psi_{x_1}(x), \cdots, \psi_{x_l}(x), h_{x_1}(x), \cdots, h_{x_l}(x))$ is the required $C^\infty\mathfrak{S}$-G imbedding. □

Here are two natural open questions.

PROBLEM A. Let M be polynomially bounded. Is a $C^\infty\mathfrak{S}$ map on an open $\mathfrak{S}$ set of class $C^\omega\mathfrak{S}$? Is a $C^\infty\mathfrak{S}$ manifold a $C^\omega\mathfrak{S}$ manifold?

Notice that if M is exponential, then $F : \mathbb{R} \longrightarrow \mathbb{R}$ in Example 2.2 (1) is a $C^\infty\mathfrak{S}$ function but not analytic. Thus the graph of F is a $C^\infty\mathfrak{S}$ manifold but not a $C^\omega\mathfrak{S}$ manifold.

PROBLEM B. Does there exist a (compact) nonaffine $C^\omega\mathfrak{S}$ manifold for any M ?

Notice that if $M = \mathfrak{R}$ (resp. $\mathbf{R}_{\mathrm{exp}}$), then Problem B is negative [15] (resp. [8]).

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