Bull. Korean Math. Soc. **36** (1999), No. 1, pp. 171–181

THE VOLTERRA-STIELTJES INTEGRAL EQUATION AND THE OPERATOR-VALUED FUNCTION SPACE INTEGRAL AS AN $\mathcal{L}(L_p,\ L_{p'})$ THEORY

KUN SIK RYU

ABSTRACT. In this note, we will prove that the operator-valued function space integral as an operator from L_p to $L_{p'}$ $(1 < p < 2)$ for certain potential functionals satisfies a Volterra- Stieltjes integral equation.

1. Introduction and preliminaries

Cameron and Storvick introduced an operator-valued function space integral in 1968 [3]. Johnson and Lapidus established the existence theorem of the operator-valued function space integral as an operator from $L_2(\mathbf{R}^N)$ to itself for certain functionals involving some Borel measures [8]. And in 1987, Lapidus proved that the integral satisfies the Schrödinger wave equation [10].

In 1992, Chang and the author established the existence theorem of the operator-valued function space integral as an operator from L_p to $L_{p'}$ $(1 < p < 2)$ for certain functionals involving some Borel measures [4]. In this note, we will prove that the integral satisfies a Volterra-Stieltjes integral equation.

Now some notations and facts which are needed in next section. Insofar as possible, we adopt the definitions and notations of [4].

Received April 13, 1998.

1991 Mathematics Subject Classification: 28C20.

Key words and phrases: operator-valued function space integral, Volterra-Stieltjes integral equation.

This paper was supported by HAN NAM University Research Fund in 1997.

A. Let $\mathbf{N}$ be the set of all natural numbers and let $\mathbf{R}$ be the set of all real numbers. Let $\mathbf{C}, \mathbf{C}_+$ and $\mathbf{C}_+^\sim$ be the set of all complex numbers, all complex numbers with positive real part and all non-zero complex numbers with non-negative real parts, respectively. Let ρ be a function on the set of all non-negative integers such that $\rho(0) = 0$ and $\rho(n) = 1$ for $n \geq 1$.

B. Given a number p such that $1 \leq p \leq +\infty$, p and p' will be always related by $\frac{1}{p} + \frac{1}{p'} = 1$. If $1 < p < 2$ is given, let α in $(1, +\infty)$ be such that $\alpha = \frac{p}{2-p}$. In our theorems, N will be a positive integer restricted so that $N < 2\alpha$. For $1 < p < 2$, let r be a real number such that $\frac{2\alpha}{2\alpha - N} < r < +\infty$. And let $\delta \equiv \frac{N}{2\alpha}$.

C. For $1 \leq p \leq +\infty$, $L_p\ (\mathbf{R}^N)$ is the space of $\mathbf{C}$ – valued Borel measurable functions ψ on $\mathbf{R}^N$ such that $|\ \psi\ |^p$ is integrable with respect to Lebesgue measure. Let $\mathcal{L}(L_p, L_{p'})$ be the space of bounded linear operators from $L_p(\mathbf{R}^N)$ into $L_{p'}(\mathbf{R}^N)$. For an operator T, T^* is a Banach adjoint of T.

D. Let $1 < p < 2$ be given. For λ in $\mathbf{C}_+^\sim$, ψ in $L_p(\mathbf{R}^N)$, ξ in $\mathbf{R}^N$ and a positive real number s, let

$$(1.1) \qquad (\mathcal{C}_{\lambda/s}\psi)(\xi) \ = \ \left(\frac{\lambda}{2\pi s}\right)^{\frac{N}{2}} \int_{\mathbf{R}^N} \psi(u)\ \exp\left(-\frac{\lambda\|u-\xi\|^2}{2s}\right) dm_L(u)$$

where if N is odd we always choose $\lambda^{-\frac{1}{2}}$ with non-negative real part and if $\mathrm{Re}\lambda \equiv 0$, the integral in (1.1) should be interpreted in the mean just as in the theory of the L_p Fourier transform. From [9], as a function of λ, $\mathcal{C}_{\lambda/s}$ is analytic in $\mathbf{C}_+$, it is strong continuous in $\mathbf{C}_+^\sim$, and it is in $\mathcal{L}(L_p, L_{p'})$ and $\|\ \mathcal{C}_{\lambda/s}\ \| \leq \left(\frac{|\lambda|}{2\pi s}\right)^\delta$. Moreover, $\mathcal{C}_{\lambda/s}^* = \mathcal{C}_{\bar\lambda/s}$ where $\mathcal{C}_{\lambda/s}^*$ means the adjoint operator of $\mathcal{C}_{\bar\lambda/s}$ and $\bar\lambda$ is the conjugate of λ. From the Chapman-Kolmogorov theorem, $\mathcal{C}_\lambda \circ \mathcal{C}_\mu \psi = \mathcal{C}_{\lambda+\mu}\psi$ whenever the integrals exist. And so we adopt just the notation as follows $\mathcal{C}_\lambda \circ 1 \circ \mathcal{C}_\mu \equiv \mathcal{C}_{\lambda+\mu}$.

E. Let $t > 0$ be given. $M(0, t)$ will denote the space of all complex Borel measures η on the interval $(0, t)$. Every measure η in $M(0, t)$ has a unique decomposition, $\eta = \mu + \nu$ into a continuous part μ and a discrete part $\nu = \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$, where $\langle \omega_p \rangle$ is a summable sequence in $\mathbf{C}$ and δ_{τ_p} is the Dirac measure [12]. And $M(0,t)^*$ will denote the subset of $M(0,t)$ which satisfies the following conditions;

(a) If μ is the continuous part of η in $M(0,t)^*$, then the Radon-Nikodym derivative $\frac{d|\mu|}{dm}$ exists and is essentially bounded, where m is the Lebesgue measure on $(0,t)$.

(b) If $\nu = \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$ is the discrete part of η in $M(0,t)^*$, then $\sum_{p=1}^{\infty} | \omega_p | \delta_{\tau_p}^{-r'\delta}$ converges.

For μ in $M(0,t)$, we define a measure $\bar{\mu}$ given by $\bar{\mu}(B) = \overline{\mu(B)}$ for all Borel subsets B of (a,b). Then for $\eta = \mu + \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$, $\bar{\eta} = \bar{\mu} + \sum_{p=1}^{\infty} \bar{\omega}_p \delta_{\tau_p}$.

F. Let $C_0[0,t] \equiv C_0$ be the space of $\mathbf{R}^N$-valued continuous functions on $[0,t]$ which vanish at 0. We consider C_0 as equipped with N-dimensional Wiener measure m_w. Let $C[0,t] \equiv C$ be the space of $\mathbf{R}^N$-valued continuous functions on $[0,t]$.

G. For $1 < p < 2$ and η in $M(0,t)$, let $L_{\alpha r:\eta}([0,t] \times \mathbf{R}^N) \equiv L_{\alpha r:\eta}$ be the space of all $\mathbf{C}$-valued Borel measurable functions θ on $[0,t] \times \mathbf{R}^N$ such that

$$(1.2) \qquad \| \theta \|_{\alpha r:\eta} = \left\{ \int_{(0,t)} \| \theta(s,\cdot) \|_{\alpha}^{r} \, d| \eta | (s) \right\}^{\frac{1}{r}} \text{ is finite.}$$

H. Let $1 < p < 2$ be given and θ be in $L_\alpha(\mathbf{R}^N)$. From Lemma 1.3 in [9], the function $M_\theta : L_{p'}(\mathbf{R}^N) \to L_p(\mathbf{R}^N)$ defined by $M_\theta(f) = f\theta$, is in $\mathcal{L}(L_{p'}, L_p)$ and $\| M_\theta \| \leq \| \theta \|_\alpha$. Moreover, $M_\theta^* = M_{\bar{\theta}}$. It will be convenient to let $\theta(s)$ denote $M_{\theta(s,\cdot)}$ for θ in $L_{\alpha r:\eta}$.

Let $\theta_1, \theta_2, \cdots, \theta_{m-1}$ be in $L_\alpha(\mathbf{R}^N)$, ψ in $L_p(\mathbf{R}^N)$ and $0 < s_1 < s_2 <$

$\cdots < s_m < t$. From the Wiener integral formula [13],

(1.3)

$$\int_{C_0} \theta_1(x(s_1))\theta_2(x(s_2)) \cdots \theta_{m-1}(x(s_{m-1}))\psi(x(s_m))\, dm_w(x)$$
$$= \left[\{\mathcal{C}_{1/s_1} \circ \theta_1(s_1) \circ \cdots \circ \mathcal{C}_{1/(s_{m-1}-s_{m-2})} \circ \theta_{m-1}(s_{m-1}) \circ \mathcal{C}_{1/(s_m-s_{m-1})}\}\psi\ \right](0).$$

I. Let $0 < k < 1$ be given and m be in $\mathbf{N}$. For $0 < s_1 < s_2 < \cdots < s_m < b$,

$$(1.4) \qquad \int_a^b \int_a^{s_m} \cdots \int_a^{s_1} \{(s_1 - a)(s_2 - s_1) \cdots (b - s_m)\}^{-k}\, ds_1 \cdots ds_m$$
$$= \frac{(b-a)^{m-(m+1)k}\{\Gamma(1-k)\}^{m+1}}{\Gamma((m+1)(1-k))},$$

where Γ is the gamma function.
Throughout this paper, this values is denoted by $E(a, b; m, k)$.

J. Let $1 < p < 2$ be given. Let F be a functional on $C[0, t]$. Given $\lambda > 0$, ψ in $L_p(\mathbf{R}^N)$ and ξ in $\mathbf{R}^N$, let

$$(1.5) \qquad [I_\lambda(F)\psi](\xi) = \int_{C_0} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\, dm_w(x).$$

If for m_L–a.e. ξ in $\mathbf{R}^N$, $[I_\lambda(F)\psi](\xi)$ exists in $L_{p'}(\mathbf{R}^N)$ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L}(L_p, L_{p'})$, we say that the operator-valued function space integral $I_\lambda(F)$ exists for λ. Suppose there exists λ_0 $(0 < \lambda_0 \le +\infty)$ such that $I_\lambda(F)$ exists for all $0 < \lambda < \lambda_0$ and there exists an $\mathcal{L}(L_p, L_{p'})$–valued function which is analytic in $\mathbf{C}_{+,\lambda_0} \equiv \mathbf{C}_+ \cap \{z \in \mathbf{C} \mid\ \mid z \mid < \lambda_0\}$ and agrees with $I_\lambda(F)$ on $(0, \lambda_0)$, then this $\mathcal{L}(L_p, L_{p'})$–valued function is called the operator-valued function space integral of F associated with λ and in this case, we say that $I_\lambda(F)$ exists for λ in $\mathbf{C}_{+,\lambda_0}$. If $I_\lambda(F)$ exists for λ in $\mathbf{C}_{+,\lambda_0}$ and $I_\lambda(F)$ is strongly continuous in $\mathbf{C}^{\sim}_{+,\lambda_0} \equiv \mathbf{C}^{\sim}_{+} \cap \{z \in \mathbf{C} \mid\ \mid z \mid < \lambda_0\}$, we say that $I_\lambda(F)$ exists for λ in $\mathbf{C}^{\sim}_{+,\lambda_0}$. When λ is purely imaginary, $I_\lambda(F)$ is called the (analytic) operator-valued Feynman integral of F.

It is mathematically convenient and physically natural to work with the Banach adjoint $I_\lambda(F)^*$ of $I_\lambda(F)$ rather than with $I_\lambda(F)$ itself. Note that $I_\lambda(F)^*$ depends on time t both through the definition of F and that of $I_\lambda(F)$ in (1.5). We thus set, for λ in $\mathbf{C}_+^\sim$ and $t > 0$,

$$u(t) = I_\lambda(F)^*. \tag{1.6}$$

In [4], Chang and the author proved the following facts.

THEOREM 1.1. *Let $1 < p < 2$ be given, let η be in $M(0,t)^*$, let θ be in $L_{\alpha r:\eta}$ and let $\eta = \mu + \nu$ be the decomposition of η into its continuous and discrete parts. Let*

$$F(y) = \exp\left\{\int_{(0,t)} \theta(s, y(s))\, d\eta(s)\right\} \text{ for y in C.} \tag{1.7}$$

Suppose that $\nu = \sum_{p=1}^{h} w_p \delta_{\tau_p}$ where we may assume that $0 < \tau_1 < \tau_2 < \cdots < \tau_h < t$ and $\theta(\tau_p, \cdot)$, $p = 1, 2, \cdots, h$, are essentially bounded. Then the operator $I_\lambda(F)$ exists for λ in $\mathbf{C}_+^\sim$ and for all λ in $\mathbf{C}_+^\sim$,

$$I_\lambda(F) = \sum_{n=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_0,\cdots,q_h \geq 0}} \sum_{j_1+\cdots+j_{h+1}=q_0} \frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!} \times \int_{\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}(t)} (L_0 \circ L_1 \circ \cdots L_h)\, d\prod_{i=1}^{q_0} \mu(s_i), \tag{1.8}$$

where for nonnegative integers $q_0, q_1, \cdots, q_h$ and $j_1, j_2, \cdots, j_{h+1}$,

$$\begin{aligned} &\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}(t) \\ &= \{(s_1, s_2, \cdots, s_{q_0}) \in (0,t)^{q_0} \mid 0 < s_1 < s_2 < \cdots < s_{j_1} < \tau_1 < s_{j_1+1} \\ &< \cdots < s_{j_1+j_2} < \tau_2 < \cdots < \tau_h < s_{j_1+j_2+\cdots+j_h+1} \\ &< \cdots < s_{j_1+\cdots+j_{h+1}} = s_{q_0} < t\} \end{aligned}$$

for $(s_1, s_2, \cdots, s_{q_0}) \in \triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}$ and $m \in \{0, 1, \cdots, h\}$,

$$\begin{aligned} L_m &= [\theta(\tau_m)]^{q_m} \circ \mathcal{C}_{\lambda/(s_{j_1+\cdots+j_m+1}-\tau_m)} \circ \theta(s_{j_1+\cdots+j_m+1}) \circ \mathcal{C}_{\lambda/(s_{j_1+\cdots+j_m+2}-s_{j_1+\cdots+j_m+1})} \\ &\circ \cdots \circ \theta(s_{j_1+\cdots+j_{m+1}}) \circ \mathcal{C}_{\lambda/(\tau_{m+1}-s_{j_1+\cdots+j_{m+1}})} \end{aligned}$$

and the integral is the Bochner integral [5].
(*We use the conventions* $\tau_0 = 0$, $\tau_{h+1} = t$ *and* $[\theta(\tau_0)]^{q_0} = 1$, *an identity map on* $L_{p'}(\mathbf{R}^N)$). *Moreover, for all* λ *in* $\mathbf{C}_+^{\sim}$,

$$(1.9) \qquad \| I_\lambda(F) \|$$

$$\leq \sum_{n=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_0,\cdots,q_h \geq 0}} \sum_{j_1+\cdots+j_{h+1}=q_0} \frac{| w_1 |^{q_1} \cdots | w_h |^{q_h}}{q_1! \cdots q_h!}$$

$$\times (q_0!)^{-1/r} \left(\frac{| \lambda |}{2\pi}\right)^{(q_0+h+1)\delta} \left(\frac{(q_0+h)!}{q_0!\, h!}\right)^{1/2r'}$$

$$\times \left[\prod_{l=1}^{h} \| \theta(\tau_l, \cdot) \|_\infty^{q_l - 1} \| \theta(\tau_l, \cdot) \|_\alpha^{\rho(q_l)}\right] (\operatorname{ess\,sup} d | \mu | / dm)^{1/2r'}$$

$$\times (\| \theta \|_{\alpha r:\mu})^{q_0} \left[\sum_{j_1+\cdots+j_{h+1}=q_0} \left\{ \prod_{l=0}^{h} E(\tau_l, \tau_{l+1}; j_{l+1}; r'\delta) \right\}^{2/r'}\right]^{1/2}.$$

REMARK 1.2. (1) In the inequality (1.9), we can show the convergence of the summation by the essentially same method as in the proof of Theorem 2 in [1].
(2) From [7], we have

$$(1.10) \qquad u(t) = I_\lambda(F)^* = \sum_{n=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_0,\cdots,q_h \geq 0}} \sum_{j_1+\cdots+j_{h+1}=q_0} \frac{\overline{w_1}^{q_1} \cdots \overline{w_h}^{q_h}}{q_1! \cdots q_h!}$$

$$\times \int_{\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}(t)} L_h^* \circ \cdots L_1^* \circ L_0^* \, d \prod_{u=1}^{q_0} \bar{\mu}(s_u).$$

2. The main theorems

In this section, we will prove that $u(t) = I_\lambda^*(F)$ satisfies a Volterra-Stieltjes integral equation.

For v in $(\tau_h, t]$ and non-negative integers $q_0, q_1, \cdots, q_h, j_1, j_2, \cdots, j_{h+1}$ such that $q_0 + \cdots + q_h = n$ and $j_1 + \cdots + j_{h+1} = q_0$, we set

$$(2.1)\qquad u_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(v) = \int_{\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}(v)} L_h^* \circ L_{h-1}^* \circ \cdots \circ L_0^* \, d\prod_{i=1}^{q_0} \bar{\mu}(s_i).$$

and

$$(2.2)\qquad w_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(v) = \int_{\tau_h}^{v} \mathcal{C}_{\bar{\lambda}/(v-s)} \circ \bar{\theta}(s) \circ u_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(s)\, d\bar{\mu}(s).$$

THEOREM 2.1. *For v in $(\tau_h, t]$, we have*

$$(2.3)\qquad w_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(v) = u_{q_0+1,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}+1}(v).$$

Proof. Let v be in $(\tau_h, t]$ and let $q_0, q_1, \cdots, q_h$ and $j_1, j_2, \cdots, j_{h+1}$ be given non-negative integers. Suppose k is the least number such that q_i is not a zero. Then

$$(2.4)\qquad w_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(v)$$

$$\overset{(1)}{=} \int_{\tau_h}^{v} \mathcal{C}_{\bar{\lambda}/(v-s_{q_0+1})} \circ \bar{\theta}(s_{q_0+1}) \Big[\int_{\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}(s_{q_0+1})} (L_h^* \circ L_{h-1}^* \circ \cdots \circ L_0^*)\, d\prod_{i=1}^{q_0} \bar{\mu}(s_i) \Big] d\bar{\mu}(s_{q_0+1})$$

$$\overset{(2)}{=} \int_{\tau_h}^{v} \mathcal{C}_{\bar{\lambda}/(v-s_{q_0+1})} \circ \bar{\theta}(s_{q_0+1}) \Big[\int_{\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}}(s_{q_0+1})} \mathcal{C}_{\bar{\lambda}/(\tau_h-s_{q_0})} \circ \bar{\theta}(s_{q_0}) \circ \mathcal{C}_{\bar{\lambda}/(s_{j_1+j_2+\cdots+j_h+1}-s_{j_1+j_2+\cdots+j_h})}$$

$$\circ\, \bar{\theta}(s_{j_1+j_2+\cdots+j_h}) \circ \cdots \circ \mathcal{C}_{\bar{\lambda}/(s_{j_1+j_2+\cdots+j_k+2}-s_{j_1+j_2+\cdots+j_k+1})}$$

$$\circ\, \bar{\theta}(s_{j_1+j_2+\cdots+j_k+1}) \circ L_k^* \circ L_{k-1}^* \circ \cdots \circ L_0^* \, d\prod_{i=1}^{q_0} \bar{\mu}(s_i) \Big] d\bar{\mu}(s_{q_0+1})$$

$$\stackrel{(3)}{=} \int_{\triangle_{q_0,\cdots,q_h;j_1,\cdots,j_{h+1}+1}(v)} \mathcal{C}_{\bar{\lambda}/(v-s_{q_0+1})} \circ \bar{\theta}(s_{q_0+1})$$

$$\circ\, \mathcal{C}_{\bar{\lambda}/(\tau_h - s_{q_0})} \circ \bar{\theta}(s_{q_0}) \circ \mathcal{C}_{\bar{\lambda}/(s_{j_1+j_2+\cdots+j_h+1} - s_{j_1+j_2+\cdots+j_h})}$$

$$\circ\, \bar{\theta}(s_{j_1+j_2+\cdots+j_h}) \circ \cdots \circ \mathcal{C}_{\bar{\lambda}/(s_{j_1+j_2+\cdots+j_k+2} - s_{j_1+j_2+\cdots+j_k+1})}$$

$$\circ\, \bar{\theta}(s_{j_1+j_2+\cdots+j_k+1}) \circ L_k^* \circ L_{k-1}^* \circ \cdots \circ L_0^* \, d \prod_{i=1}^{q_0+1} \bar{\mu}(s_i)$$

$$\stackrel{(4)}{=} \int_{\triangle_{q_0+1,\cdots,q_h;j_1,\cdots,j_{h+1}+1}(v)} (L_h^* \circ L_{h-1}^* \circ \cdots \circ L_0^*) \, d \prod_{i=1}^{q_0+1} \bar{\mu}(s_i)$$

$$\stackrel{(5)}{=} u_{q_0+1,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}+1}(v). \qquad \square$$

Steps (1) and (5) follow from the definitions of u, w in (2.1) and (2.2) respectively. By the assumption, we have Step (2). Step (3) follows, we know that the following two facts are equivalent;

(1) s_{q_0+1} is in (τ_h, v) and $(s_1, s_2, \cdots, s_{q_0})$ is in $\triangle_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(s_{q_0+1})$,

(2) $(s_1, s_2, \cdots, s_{q_0}, s_{q_0+1})$ is in $\triangle_{q_0+1,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}+1}(v)$.

And so, by the D in section 2, we obtain Step (4).

THEOREM 2.2. *The operator-valued function u satisfies the Volterra-Stieltjes integral equation*

$$u(v) = \mathcal{C}_{\bar{\lambda}/(v-\tau_h)} \circ e^{\overline{w_h}\bar{\theta}(\tau_h)} \circ u(\tau_h) + \int_{\tau_h}^{v} \mathcal{C}_{\bar{\lambda}/(v-s)} \circ \bar{\theta}(s) \circ u(s) \, d\bar{\mu}(s) \tag{2.5}$$

for all v in (τ_h, t).

Proof. For v in (τ_h, t), let

$$w(v) = \int_{\tau_h}^{v} \mathcal{C}_{\bar{\lambda}/(v-s)} \circ \bar{\theta}(s) \circ u(s) \, d\bar{\mu}(s). \tag{2.6}$$

Then

$$
\begin{aligned}
(2.7)\quad w(v) &\overset{(1)}{=} \sum_{n=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_0,\cdots,q_h\geq 0}} \sum_{j_1+\cdots+j_{h+1}=q_0} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \\
&\qquad \times \int_{\tau_h}^{v} \mathcal{C}_{\bar{\lambda}/(v-s)} \circ \bar{\theta}(s) \circ u_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(s)\, d\bar{\mu}(s). \\
&\overset{(2)}{=} \sum_{n=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_0,\cdots,q_h\geq 0}} \sum_{j_1+\cdots+j_{h+1}=q_0} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \\
&\qquad \times w_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}}(v) \\
&\overset{(3)}{=} \sum_{n=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_0,\cdots,q_h\geq 0}} \sum_{j_1+\cdots+j_{h+1}=q_0} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \\
&\qquad \times u_{q_0+1,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}+1}(v) \\
&\overset{(4)}{=} \sum_{n=0}^{\infty}\sum_{q_0=0}^{n} \sum_{q_1+\cdots+q_h=n-q_0} \sum_{j_{h+1}=0}^{q_0} \sum_{j_1+\cdots+j_h=q_0-j_{h+1}} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \\
&\qquad \times u_{q_0+1,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}+1}(v) \\
&\overset{(5)}{=} \sum_{n^*=1}^{\infty}\sum_{q_0^*=1}^{n^*} \sum_{q_1+\cdots+q_h=n^*-q_0^*} \sum_{j_{h+1}^*=0}^{q_0^*} \sum_{j_1+\cdots+j_h=q_0^*-j_{h+1}^*} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \\
&\qquad \times u_{q_0^*,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}^*}(v) \\
&\overset{(6)}{=} \sum_{n^*=0}^{\infty}\sum_{q_0^*=1}^{n^*} \sum_{q_1+\cdots+q_h=n^*-q_0^*} \sum_{j_{h+1}^*=0}^{q_0^*} \sum_{j_1+\cdots+j_h=q_0^*-j_{h+1}^*} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \\
&\qquad \times u_{q_0^*,q_1,\cdots,q_h;j_1,j_2,\cdots,j_{h+1}^*}(v).
\end{aligned}
$$

Step (1) follows from Theorem 1.1 and the the bounded convergence theorem. From (2,2), we have Step (2). By Theorem 2.1, we obtain Step (3). Step (4) is trivial. Putting $n^* = n+1, q_0^* = q_0+1$ and $j_{h+1}^* = j_{h+1}+1$, we have $n^* - q_0^* = n - q_0$, and $q_0^* - j_{h+1}^* = q_0 - j_{h+1}$, and so Step (5) holds. If $n^* = 0$, then $q_0^* = 0$, which is impossible, so we have Step (6). Thus, we have

$$u(v) - w(v) = \sum_{n=0}^{\infty} \sum_{q_0+\cdots+q_h=n} \sum_{j_1+\cdots+j_h=q_0} \frac{\overline{w_1}^{q_1}\cdots\overline{w_h}^{q_h}}{q_1!\cdots q_h!} \times u_{q_0,q_1,\cdots,q_h;j_1,j_2,\cdots,j_h,0}(v). \tag{2.8}$$

Moreover, $\triangle_{q_0,q_1,\cdots,q_h;j_1,\cdots,j_h,0}(v) = \triangle_{q_0,q_1,\cdots,q_h;j_1,\cdots,j_h}(\tau_h)$ and on $\triangle_{q_0,q_1,\cdots,q_h;j_1,\cdots,j_h,0}(v)$,

$$u_{q_0,q_1,\cdots,q_h;j_1,\cdots,j_h,0}(v) = C_{\bar\lambda/(v-\tau_h)} \circ [\bar\theta(\tau_h)]^{q_h} \circ u_{q_0,q_1,\cdots,q_{h-1};j_1,j_2,\cdots,j_h}(\tau_h). \tag{2.9}$$

Therefore

$$\begin{aligned} u(v) - w(v) &= C_{\bar\lambda/(v-\tau_h)} \circ \left(\sum_{n=0}^{\infty}\sum_{q_h=0}^{n} \frac{[\overline{w_h}\bar\theta(\tau_h)]^{q_h}}{q_h!} \circ \sum_{q_1+\cdots+q_{h-1}=n-q_0} \right. \\ &\quad \left. \cdot \frac{\overline{w_1}^{q_1}\cdots\overline{w_{h-1}}^{q_{h-1}}}{q_1!\cdots q_{h-1}!} \sum_{j_1+\cdots+j_h=q_0} u_{q_0,q_1,\cdots,q_{h-1};j_1,j_2,\cdots,j_h}(\tau_h) \right). \\ &= C_{\bar\lambda/(v-\tau_h)} \circ \exp[\overline{w_h}\bar\theta(\tau_h)] \circ u(\tau_h), \qquad \text{as desired.} \end{aligned} \tag{2.10}$$

References

[1] J. M. Ahn, K. S. Chang, J. G. Kim, J. W. Ko, K. S. Ryu, *Existence theorem of an operator-valued Feynman integral as an $\mathcal{L}(L_1, C_0)$ theory*, Bull. Korean Math. Soc. **34** (1997), 317-334.

[2] R. H. Cameron, *A family of integrals serving to connect the Wiener and Feynman integrals*, J. Math. and Phys. **39** (1960), 126-139.

[3] R. H. Cameron, D. A. Storvick, *An operator-valued function space integral and a related integral equation*, J. Math. and Mech. **18** (1968), 517-552.
[4] K. S. Chang and K. S. Ryu, *Analytic operator-valued function space integrals as an $\mathcal{L}(L_p, L_{p'})$ theory*, Trans. Amer. Math. Soc. **330** (1992), 697-709.
[5] K. S. Chang, K. S. Choi, J. A. Lim and K. S. Ryu, *Uniform continuity of Schrödinger type operators and the operator-valued function space integral*, to appear.
[6] R. P. Feynman, *Space-time approach to non-relative quantum mechanics*, Rev. Mod. Phys. **20** (1948), 367-387.
[7] E. Hille and R. S. Phillips, *Functional analysis and Semi-groups*, A. M. S. Collq. Publ., Vol. 31, 1957.
[8] G. W. Johnson and M. L. Lapidus, *Generalized Dyson series, Generalized Feynman Diagrams, the Feynman integral and Feynman's operational Calculus*, Mem. Amer. Math. Soc., Vol. 62, No. 351, 1986.
[9] G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral; An $\mathcal{L}(L_p, L_{p'})$ theory*, Nagoya Math. J. **60** (1976), 93-137.
[10] M. L. Lapidus, *The Feynman formula with a Lebesgue-Stieltjes measure and Feynman's operational Calculus*, Stud. Appl. Math. **76** (1987), 93-132.
[11] E. Nelson, *Feynman integral and the Schrödinger equation*, J. Math. Phys. **5** (1964), 332-343.
[12] M. Reed and B. Simon, *Methods of Mathematical Physics*, Vol. I, II, Rev. and enl. ed., Academic Press, New York, 1980.
[13] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.
[14] K. Yosida, *Functional Analysis*, Springer-Verlag, New York, 1974.

DEPARTMENT OF MATHEMATICS, HAN NAM UNIVERSITY, TAEJON 306-791, KOREA