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LARGE SIEVE FOR GENERALIZED TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. Generalized nonnegative trigonometric polynomials are defined as the products of nonnegative trigonometric polynomials raised to positive real powers. The generalized degree can be defined in a natural way. We improve and extend the large sieve involving pth powers of trigonometric polynomials so that it holds for generalized trigonometric polynomials.

1. Introduction

The large sieve is an inequality of the following form. See [9, Theorem 3, p. 559], but note the different notation.

For any trigonometric polynomial S_N of degree at most N,

$$S_N(\tau) = \sum_{k=-N}^{N} a_k e^{ik\tau}, \qquad \tau \in [0, 2\pi),$$

$$\sum_{j=1}^{M} |S_N(\tau_j)|^2 \leq \left(\frac{N}{\pi} + \delta^{-1}\right) \int_0^{2\pi} |S_N(\theta)|^2 d\theta \,, \tag{1.1}$$

whenever $0 \leq \tau_1 < \tau_2 < \cdots < \tau_M \leq 2\pi$ and

$$\delta = \min\{\tau_2 - \tau_1, \cdots, \tau_M - \tau_{M-1}, 2\pi - (\tau_M - \tau_1)\} > 0 \,.$$

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The large sieve originates in a short paper of Ju. V. Linnik [7]. In number theory, the large sieve plays an important role in partial solution of Goldbach Conjecture, which asserts that every even integer greater than 2 is the sum of two primes. Using the large sieve, Rényi [11], [12] showed that every large even integer $2N$ can be expressed in the form $2N = p + R_k$, where p is prime and R_k has at most k prime factors. Later Chen [1], [2] has shown that one can take $k = 2$.

The large sieve is useful in trigonometric interpolation and approximation. In [8, Theorem 2, p. 533], Lubinsky, Máté, and Nevai extended (1.1) to sums involving pth powers as follows.

Let $0 < p < \infty$. Let Ψ be convex, nonnegative, and nondecreasing in $[0, \infty)$. Then for any trigonometric polynomial S_N of degree at most $N \in \mathbb{N}$,

$$\sum_{j=1}^{M} \Psi(|S_N(\tau_j)|^p) \le \left(\frac{N}{\pi} + \delta^{-1}\right) \int_0^{2\pi} \Psi(|S_N(u)|^p (p+1)e/2)du \,, \tag{1.2}$$

whenever $0 \le \tau_1 < \tau_2 < \cdots < \tau_M \le 2\pi$ and

$$\delta = \min\{\tau_2 - \tau_1, \cdots, \tau_M - \tau_{M-1}, 2\pi - (\tau_M - \tau_1)\} > 0 \,.$$

REMARK. Note the differences between (1.2) and Theorem 2 of [8, p. 533]. The factor $(2n + \delta^{-1})(2\pi)^{-1}$ in [8, Theorem 2, p. 533] should be replaced by $\left(\frac{n}{\pi} + \delta^{-1}\right)$.

The purpose of this paper is to generalize (1.2) so that it holds for generalized trigonometric polynomials as well and to improve the inequality (1.2) using this generalization.

The function

$$f(z) = |\omega| \prod_{j=1}^{m} \left|\sin\left(\frac{z - z_j}{2}\right)\right|^{r_j}$$

with $r_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$, and $0 \ne \omega \in \mathbb{C}$ is called a generalized nonnegative trigonometric polynomial of generalized degree

$$n \stackrel{\text{def}}{=} \frac{1}{2} \sum_{j=1}^{m} r_j \,.$$

We denote by GTNP_n the set of all generalized nonnegative trigonometric polynomials of degree at most $n \in \mathbb{R}^+$.

Note that, here, $n > 0$ is not necessarily an integer. In fact, we assume throughout this paper that $n \in \mathbb{R}^+$ unless stated otherwise.

In what follows we denote by $\mathbb{T}_N$, $(N \in \mathbb{N})$, the set of all trigonometric polynomials of degree at most N.

In this paper we study generalized nonnegative trigonometric polynomials restricted to the real line. Using

$$\begin{aligned} \left|\sin\left(\frac{z-z_j}{2}\right)\right| &= \left(\sin\left(\frac{z-z_j}{2}\right)\sin\left(\frac{z-\bar{z}_j}{2}\right)\right)^{1/2} \\ &= \left(\frac{\cosh(\mathrm{Im} z_j) - \cos(z - \mathrm{Re} z_j)}{2}\right)^{1/2}, \\ & \quad z \in \mathbb{R}, \end{aligned}$$

we can easily check that when $f \in \mathrm{GTNP}_n$ is restricted to the real line, then it can be written as

$$f = \prod_{j=1}^{m} P_j^{r_j/2}, \quad 0 \le P_j \in \mathbb{T}_1\,,\ r_j \in \mathbb{R}^+, \quad \sum_{j=1}^{m} r_j \le 2n\,,$$

which is the product of nonnegative trigonometric polynomials raised to positive real powers. This explains the name *generalized nonnegative trigonometric polynomials.* Many properties of generalized nonnegative trigonometric polynomials were investigated in a series of papers (cf. [3–6]).

The rest of this paper is organized as follows. In Section 2, we state our results. We present the proof of theorems in Section 3.

2. Results

In this paper we denote by $D_N(t) = \sum_{k=-N}^{N} e^{ikt}$, $(N \in \mathbb{N})$, the Nth Dirichlet kernel and, for each $n \in \mathbb{R}$, the symbol $[n]$ denotes the integer part of n.

Now we state our results.

THEOREM 2.1. *Let $0 < r < \infty$. Let Ψ be convex, nonnegative, and nondecreasing in $[0, \infty)$. Then for all $f \in \mathrm{GTNP}_n$, $n \in \mathbb{R}^+$,*

(2.1)
$$\Psi(f(\tau)) \leq (2\pi)^{-1}(2N+1)^{-1} \int_0^{2\pi} \Psi(f(u)(r+1)e/2) D_N^2(\tau - u)\, du\,,$$
$$\tau \in [0, 2\pi]\,,$$

where $N = \left[\frac{n}{2r} + \frac{1}{2}\right]$.

The following is an analogue of (1.2) for generalized trigonometric polynomials. Note that if $f \in \mathrm{GTNP}_n$ then $f^p \in \mathrm{GTNP}_{np}$, hence, we don't have to keep pth powers in the following.

THEOREM 2.2. *Let $0 < r < \infty$. Let Ψ be convex, nonnegative, and nondecreasing in $[0, \infty)$.*
Let
$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_M \leq 2\pi$$
and
$$\delta = \min\{\tau_2 - \tau_1, \cdots, \tau_M - \tau_{M-1}, 2\pi - (\tau_M - \tau_1)\} > 0\,.$$
Then for all $f \in \mathrm{GTNP}_n$, $n \in \mathbb{R}^+$,

$$\sum_{j=1}^{M} \Psi(f(\tau_j)) \leq \left(\frac{N}{\pi} + \delta^{-1}\right) \int_0^{2\pi} \Psi(f(u)(r+1)e/2)\, du\,, \tag{2.2}$$

where $N = \left[\frac{n}{2r} + \frac{1}{2}\right]$.

Using Theorem 2.2 we improve the inequality (1.2) as follows.

THEOREM 2.3. *Let $0 < p < \infty$. Let Ψ be convex, nonnegative, and nondecreasing in $[0, \infty)$. Then for any trigonometric polynomial S_N of degree at most $N \in \mathbb{N}$,*

(2.3)
$$\sum_{j=1}^{M} \Psi(|S_N(\tau_j)|^p) \leq \left(\frac{N+1}{2\pi} + \delta^{-1}\right) \int_0^{2\pi} \Psi(|S_N(u)|^p (p+1)e/2) du\,,$$

whenever $0 \leq \tau_1 < \tau_2 < \cdots < \tau_M \leq 2\pi$ *and*
$$\delta = \min\{\tau_2 - \tau_1, \cdots, \tau_M - \tau_{M-1}, 2\pi - (\tau_M - \tau_1)\} > 0\,.$$

REMARK. Inequality (2.3) clearly gives better upper bounds than (1.2) for $N = 2, 3, 4, \cdots$.

3. Proofs

In this section we give the proof of theorems in Section 2. To prove Theorem 2.1 we need the following lemma.

LEMMA 3.1. *Let* $f \in \mathrm{GTNP}_n$. *Then*

$$f(x) \leq \frac{(1+n)e}{4\pi} \int_0^{2\pi} f(\theta)\, d\theta \;, \quad x \in [0, 2\pi]. \tag{3.1}$$

Before we prove Lemma 3.1 we state the following theorem which will be used later. See [10, Theorem 6, p. 148].

THEOREM (Máté and Nevai). *Let* $0 < p < \infty$. *Let* P_N *be a complex algebraic polynomial of degree at most* $N \in \mathbb{N}$ *and let* g *be analytic in* $|w| < R$, $(R > 1)$. *Then*

$$|P_N(z)|^p |g(\rho z)|^2 \leq \frac{(2+Np)e}{8\pi} \int_0^{2\pi} |P_N(e^{i\theta})|^p |g(e^{i\theta})|^2\, d\theta, \tag{3.2}$$

where z *is an arbitrary point with* $|z| = 1$ *and* $\rho = Np/(2+Np)$.

Proof of Lemma 3.1. First we prove (3.1) for trigonometric polynomials. Let $T_N, (N \in \mathbb{N})$, be a trigonometric polynomial of degree at most N. We write

$$\begin{aligned} T_N(\theta) &= a_0 + \sum_{k=1}^{N} (a_k \cos k\theta + b_k \sin k\theta) \\ &= a_0 + \sum_{k=1}^{N} \left(a_k \frac{e^{ik\theta} + e^{-ik\theta}}{2} + b_k \frac{e^{ik\theta} - e^{-ik\theta}}{2i} \right) \end{aligned}$$

in the form

$$\begin{aligned} T_N(\theta) &= \sum_{k=-N}^{N} c_k e^{ik\theta} \\ &= e^{-iN\theta} \sum_{k=-N}^{N} c_k e^{i(k+N)\theta} \\ &= e^{-iN\theta} \sum_{k=0}^{2N} d_k e^{ik\theta} . \end{aligned}$$

Define the algebraic polynomial P_{2N} of degree at most $2N$ by

$$P_{2N}(z) = d_0 + d_1 z + d_2 z^2 + \cdots + d_{2N} z^{2N} .$$

Then we obtain $|P_{2N}(e^{i\theta})| = |T_N(\theta)|$. Let $0 < p < \infty$. Applying (3.2) to P_{2N} with $g \equiv 1$ yields

(3.3) $$|T_N(x)|^p \le \frac{(1+pN)e}{4\pi} \int_0^{2\pi} |T_N(\theta)|^p \, d\theta , \quad x \in \mathbb{R},$$

for any $T_N \in \mathbb{T}_N$, $(N \in \mathbb{N})$. Now we extend (3.3) to generalized trigonometric polynomials. Let $f \in \mathrm{GTNP}_n$, $(n \in \mathbb{R}^+)$. Then f can be written as

(3.4)
$$f(x) = |\omega| \prod_{j=1}^{m} \left| \sin\left(\frac{x - z_j}{2} \right) \right|^{r_j} , \quad \omega \ne 0,\ z_j \in \mathbb{C},\ r_j \in \mathbb{R}^+,\ \sum_{j=1}^{m} r_j \le 2n .$$

First assume that $r_j \in \mathbb{Q}$ for $1 \le j \le m$ in (3.4). Then $r_j = q_j/q$ for some positive integers q_j and q. Define

$$\begin{aligned} T(x) &= |\omega|^{2q} \prod_{j=1}^{m} \left(\sin\left(\frac{x - z_j}{2} \right) \sin\left(\frac{x - \bar{z}_j}{2} \right) \right)^{q_j} \\ &= |\omega|^{2q} \prod_{j=1}^{m} \left| \sin\left(\frac{x - z_j}{2} \right) \right|^{2q_j} . \end{aligned}$$

Then T is a trigonometric polynomial of degree at most $2qn$ and $|T(x)|^{1/(2q)} = f(x)$. Applying (3.3) to T with $1/(2q)$ instead of p, we have

$$f(x) \le \frac{(1+n)e}{4\pi} \int_0^{2\pi} f(\theta)\, d\theta\,,$$

for all $f \in \mathrm{GTNP}_n$ with $r_j \in \mathbb{Q}$ in its representation (3.4). In the case of positive real exponents r_j in (3.4), we can obtain (3.1) using the above inequality and approximation. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $f \in \mathrm{GTNP}_n$, $(n \in \mathbb{R}^+)$. Let $0 < r < \infty$ and let $N = \left[\frac{n}{2r} + \frac{1}{2}\right]$. Let

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}\,, \qquad x \in [0, 2\pi).$$

Note that $fD_N^2 \in \mathrm{GTNP}_{n+2N}$. If we apply Lemma 3.1 to $f(x)D_N^2(\tau - x)$ with τ fixed, we have

$$f(x)D_N^2(\tau - x) \le \frac{(1+n+2N)e}{4\pi} \int_0^{2\pi} f(u)D_N^2(\tau - u)\, du\,. \tag{3.5}$$

Since $\frac{n}{2r} - \frac{1}{2} < N$, we have $n < 2rN + r$, so that

$$1 + n + 2N < (r+1)(2N+1)\,,$$

therefore,

$$f(x)D_N^2(\tau - x) \le \frac{(r+1)(2N+1)e}{4\pi} \int_0^{2\pi} f(u)D_N^2(\tau - u)\, du\,. \tag{3.6}$$

Setting $x = \tau$ in (3.6), and using

$$D_N(0) = 2N + 1\,,$$

we have

$$f(\tau) \le \frac{(r+1)e}{4\pi(2N+1)} \int_0^{2\pi} f(u)D_N^2(\tau - u)du\,, \qquad x \in [0, 2\pi).$$

Since

$$\int_0^{2\pi} D_N^2(\tau - u)du = 2\pi(2N+1)\,,$$

we have

$$f(\tau) \le \frac{(r+1)e}{2} \, \frac{\int_0^{2\pi} f(u)D_N^2(\tau - u)\, du}{\int_0^{2\pi} D_N^2(\tau - u)\, du}\,.$$

Now suppose that Ψ is convex, nonnegative, and nondecreasing in $[0,\infty)$. Then

$$\begin{aligned}\Psi(f(\tau)) &\leq \Psi\left(\frac{(r+1)e}{2}\,\frac{\int_0^{2\pi} f(u)D_N^2(\tau-u)\,du}{\int_0^{2\pi} D_N^2(\tau-u)\,du}\right)\\ &\leq \frac{1}{2\pi(2N+1)}\int_0^{2\pi}\Psi(f(u)(r+1)e/2)D_N^2(\tau-u)\,du\,,\\ &\qquad\qquad \tau\in[0,2\pi)\,,\end{aligned}$$

by Jensen' inequality (Zygmund [13, p. 24]), which completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let $0<r<\infty$. Let Ψ be convex, nonnegative, and nondecreasing in $[0,\infty)$.
Let

$$0\leq\tau_1<\tau_2<\cdots<\tau_M\leq 2\pi$$

and

$$\delta=\min\{\tau_2-\tau_1,\cdots,\tau_M-\tau_{M-1},2\pi-(\tau_M-\tau_1)\}>0\,.$$

Let $f\in \mathrm{GTNP}_n$, $n\in\mathbb{R}^+$, and let $N=\left[\frac{n}{2r}+\frac{1}{2}\right]$. Applying (1.1) to $D_N(\tau-u)$, we have

$$\sum_{j=1}^{M} D_N^2(\tau_j-u)\leq 2\pi(2N+1)\left(\frac{N}{\pi}+\delta^{-1}\right),\ \text{ for } u\in[0,2\pi),$$

thus, Theorem 2.2 follows by Theorem 2.1 and the above inequality. □

Proof of Theorem 2.3. Let $0<p<\infty$. Let Ψ be convex, nonnegative, and nondecreasing in $[0,\infty)$.
Let

$$0\leq\tau_1<\tau_2<\cdots<\tau_M\leq 2\pi$$

and

$$\delta=\min\{\tau_2-\tau_1,\cdots,\tau_M-\tau_{M-1},2\pi-(\tau_M-\tau_1)\}>0\,.$$

Let S_N be a trigonometric polynomial of degree at most $N\in\mathbb{N}$. Then $|S_N|^p$ is a generalized trigonometric polynomial of degree at most Np. Applying (2.2) to $|S_N|^p\in \mathrm{GTNP}_{Np}$ with $r=p$ yields Theorem 2.3. □

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