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NILPOTENCY CLASSES OF RIGHT NILPOTENT CONGRUENCES

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ABSTRACT. It is known that a right nilpotent congruence β on a finite algebra $\mathbf{A}$ is also left nilpotent [3]. The question on whether the left nilpotency class of β is less than or equal to the right nilpotency class of β is still open.

In this paper we find an upper limit for the left nilpotency class of β.

In addition, under the assumption that $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$, we show that $(\beta]^k = [\beta)^k$ for all $k \geq 1$. Thus the left and right nilpotency classes of β are the same in this case.

1. Introduction

The commutator operation on the lattice of congruence relations on algebras has found its deepest applications in the study of congruence modular varieties [1]. The list of problems that have been solved using commutators are too long to mention here. For the nonmodular case the usefulness of commutators is severely limited because in nonmodular varieties the commutators no longer enjoy some nice properties such as commutativity and complete additivity.

However, in [2] D. Hobby and R. McKenzie obtain various results concerning commutators for *finite algebras* in nonmodular varieties using the tame congruence theory developed by them. In [3] K. Kearnes devises a new method, using tame congruence theory heavily, of overcoming the lack of those nice properties of commutators in finite algebras in nonmodular varieties, which he calls *annihilation of tame quotients.* His

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method greatly expands the applicability of commutators for algebras that does not generate modular varieties.

One of the nicest properties of the commutators in a modular variety $\mathcal{V}$ is the following equivalence of the 4 conditions

$$C(\beta,\theta;\delta) \Leftrightarrow \delta \geq [\beta,\theta] \Leftrightarrow C(\theta,\beta;\delta) \Leftrightarrow \delta \geq [\theta,\beta]$$

where β, θ and δ are any congruences on any algebra in $\mathcal{V}$. In general, for algebras in nonmodular varieties, these 4 conditions are not equivalent to each other.

In [3] K. Kearnes examines the relationship among the 4 conditions: $C(\beta,\theta;\delta)$, $\delta \geq [\beta,\theta]$, $C(\theta,\beta;\delta)$, and $\delta \geq [\theta,\beta]$ *for the case when* $\langle\delta,\theta\rangle$ *is a tame quotient in a finite algebra* $\mathbf{A}$ and β is any congruence on $\mathbf{A}$. It is obvious that $C(\beta,\theta;\delta) \Rightarrow \delta \geq [\beta,\theta]$ and $C(\theta,\beta;\delta) \Rightarrow \delta \geq [\theta,\beta]$. But in general all other implications fail. Kearnes shows that under some moderate assumptions some of the nonobvious implications hold.

He obtains various interesting results using the nonobvious implications among the 4 conditions. One of the results is that right nilpotent congruences in finite algebras are also left nilpotent. But left nilpotent congruences are not necessarily right nilpotent, and for infinite algebras right nilpotence does not imply left nilpotence.

One of the problems he poses in [3] is whether the left nilpotency class is less than or equal to the right nilpotency class for a right nilpotent congruence β in a finite algebra $\mathbf{A}$. We do not settle this problem here. However, we find an upper bound on the left nilpotency class of β. The upper bound we propose is rather rough though. In addition we find the answer to above problem to be affirmative *provided that the type set of* $\mathbf{A}$ *omits the unary type.* In fact the left and right nilpotency classes are the same in this case.

As a matter of fact the condition of $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$ is strong enough to imply that left nilpotent congruences are also right nilpotent. This fact is essentially included in the work of Kearnes' in [3] although he does not explicitly say so. Nevertheless, even under the assumption $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$, it is not immediate to get "left nilpotency class = right nilpotency class."

We assume that the reader has a nodding knowledge in universal algebra, commutator theory and tame congruence theory. Our reference for these subjects are [4], [1] and [2].

2. Nilpotent congruences

We use the following notation. Let α and β be congruences of an algebra $\mathbf{A}$.

(1) $(\alpha]^1 = [\alpha)^1 = [\alpha]^1 = \alpha$.
(2) $(\alpha, \beta]^1 = [\alpha, \beta)^1 = [\alpha, \beta]$.
(3) $(\alpha]^{n+1} = [\alpha, (\alpha]^n]$,
(4) $[\alpha)^{n+1} = [[\alpha)^n, \alpha]$,
(5) $[\alpha]^{n+1} = [[\alpha]^n, [\alpha]^n]$,
(6) $(\alpha, \beta]^{n+1} = [\alpha, (\alpha, \beta]^n]$,
(7) $[\alpha, \beta)^{n+1} = [[\alpha, \beta)^n, \beta]$.

For $n \in \mathbb{N}$, if $(\alpha]^{n+1} = 0_A$, then we say that α is *n-step left nilpotent.* α is *left nilpotent* iff it is n-step left nilpotent for some n. The *left nilpotency class* of α is the natural number n such that α is n-step nilpotent but not $(n-1)$-step nilpotent. Similarly we define the *right nilpotency (class).*

Recall that α is said to be solvable if $[\alpha]^n = 0_A$ for some n. Note that nilpotent congruences are solvable, but not vice versa.

THEOREM 1 (K. Kearnes). *Let $\mathbf{A}$ be a finite algebra. Suppose that $\langle\delta, \theta\rangle$ is a tame quotient of $\mathbf{A}$, U is a $\langle\delta, \theta\rangle$-minimal set with body B and tail T, and β is any congruence on $\mathbf{A}$. Consider the following four conditions:*

(a) $C(\beta, \theta; \delta)$,
(b) $\delta \geq [\beta, \theta]$,
(c) $C(\theta, \beta; \delta)$,
(d) $\delta \geq [\theta, \beta]$.

We have

(1) If typ$(\delta, \theta) \in \{3, 4, 5\}$, then $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$ holds. If $(\beta]^k|_U \subseteq B^2 \cup T^2$ for some $k \geq 1$, then they are all equivalent.
(2) If typ$(\delta, \theta) \in \{2\}$, then $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$ holds. If $(\beta]^k|_U \subseteq B^2 \cup T^2$ for some $k \geq 1$, then they are all equivalent.
(3) If typ$(\delta, \theta) \in \{1\}$ and if $(\beta]^k|_U \subseteq B^2 \cup T^2$ for some $k \geq 1$, then $(c) \Rightarrow (d) \Rightarrow (a) \Leftrightarrow (b)$ holds.

Proof. See [3]. □

COROLLARY 1 (K. Kearnes). *Any right nilpotent congruence on a finite algebra is also left nilpotent.*

Proof. See [3]. □

Following is a useful lemma that is easily proved.

LEMMA 1. *If β is a right nilpotent congruence on a finite algebra* **A** *and if $\langle \delta, \theta \rangle$ is a tame quotient in* Con **A**, *then we have the implication*

$$\delta \geq [\theta, \beta] \Rightarrow \delta \geq [\beta, \theta].$$

Proof. By corollary 1 there exists an integer k such that $(\beta]^k|_U = 0_A$. Hence

$$(\beta]^k|_U = 0_A|_U \subseteq B^2 \cup T^2.$$

Therefore we can apply lemma 1 to get the desired implication. □

Using above lemma we can obtain an upper limit for the left nilpotency class of a right nilpotent congruence on a finite algebra as in the following theorem.

THEOREM 2. *Let β be a right nilpotent congruence on a finite algebra* **A**. *Let $n \in \mathbb{N}$ be such that there exists a sequence*

$$\beta = \gamma_0 \succ \gamma_1 \succ \cdots \succ \gamma_n = 0_A$$

of coverings of congruences such that $[\beta)^k \in \{\gamma_0, \cdots, \gamma_n\}$ for all $k \geq 1$. Then $(\beta]^{n+1} = 0_A$: i.e., the left nilpotency class of β is less than or equal to n.

Proof. We will show that $\gamma_i \geq (\beta]^{i+1}$ for each $i \leq n$ by induction on i. That will suffice. For $i = 0$ there is nothing to prove. Assume $\gamma_i \geq (\beta]^{i+1}$. We need to show $\gamma_{i+1} \geq (\beta]^{i+2}$.

Choose $k \geq 1$ so that $[\beta)^k \geq \gamma_i \succ \gamma_{i+1} \geq [\beta)^{k+1}$. Then

$$\gamma_{i+1} \geq [\beta)^{k+1} = [[\beta)^k, \beta] \geq [\gamma_i, \beta].$$

We apply lemma 1 to get

$$\gamma_{i+1} \geq [\beta, \gamma_i].$$

Now by the induction hypothesis $\gamma_i \geq (\beta]^{i+1}$, we see that

$$\gamma_{i+1} \geq [\beta, \gamma_i] \geq [\beta, (\beta]^{i+1}] = (\beta]^{i+2}$$

which was to be shown. □

In lemma 1 the condition of the quotient $\langle\delta,\theta\rangle$ being tame was critical. We can strengthen the lemma by replacing this tameness condition with "$\langle\delta,\theta\rangle$ is tame or $\operatorname{typ}\{\delta,\theta\} = \{\mathbf{2}\}$." Further we can prove the other direction $\delta \geq [\theta,\beta] \Leftarrow \delta \geq [\beta,\theta]$ under the assumption $\operatorname{typ}\{\delta,\theta\} = \{\mathbf{2}\}$ and $\theta \leq \beta$.

LEMMA 2. *Let $\beta \geq \theta > \delta$ be congruences of a finite algebra $\mathbf{A}$. If β is a right nilpotent congruence on $\mathbf{A}$ and if $\operatorname{typ}\{\delta,\theta\} = \{\mathbf{2}\}$, then we have the biimplication*

$$\delta \geq [\theta,\beta] \Leftrightarrow \delta \geq [\beta,\theta].$$

Proof. Let us prove ($\Rightarrow$) first. Suppose $\delta \geq [\theta,\beta]$ and assume, for the purpose of getting a contradiction, that $\delta \not\geq [\beta,\theta]$. The assumption implies the failure of $C(\beta,\theta;\delta)$, and hence there exist $(u,v) \in \beta$, $(a_i,b_i) \in \theta$ for $i = 1,\cdots,n$ and $p(x,\bar{y}) \in \operatorname{Pol}_{n+1}\mathbf{A}$ such that

$$p(u,\bar{a}) \stackrel{\delta}{\equiv} p(u,\bar{b}) \text{ and } p(v,\bar{a}) \stackrel{\theta-\delta}{\equiv} p(v,\bar{b}).$$

We are going to find $p' \in \operatorname{Pol}_{n+1}\mathbf{A}$ such that

$$p'(u,\bar{a}) = p'(v,\bar{a}) \text{ and } p'(u,\bar{b}) \stackrel{\delta}{\not\equiv} p'(v,\bar{b}). \tag{1}$$

That will suffice to show $\delta \not\geq [\theta,\beta]$.

First, choose a prime quotient $\langle\gamma,\lambda\rangle$ in the interval $I[\delta,\theta]$ so that

$$p(v,\bar{a}) \stackrel{\lambda-\gamma}{\equiv} p(v,\bar{b}).$$

Let U be any $\langle\gamma,\lambda\rangle$-minimal set with body B and tail T. Note that $\mathbf{A}|_U$ is of affine type from the hypothesis of this lemma. Without loss of generality the range of p is U, and the two elements $p(v,\bar{a})$ and $p(v,\bar{b})$ belong to B. Observe that β is left nilpotent by corollary 1. Thus $(\beta]^k|_U = 0_A|_U \subseteq B^2 \cup T^2$ for some $k \geq 1$. Now by a basic lemma of the tame congruence theory, we get $\beta|_U \subseteq B^2 \cup T^2$. (See lemma 4.27.(4).(ii) in [2].) Since $p(u,\bar{a}) \stackrel{\beta}{\equiv} p(v,\bar{a})$ and $p(u,\bar{b}) \stackrel{\beta}{\equiv} p(v,\bar{b})$ we now have all the 4 elements $p(u,\bar{a})$, $p(u,\bar{b})$, $p(v,\bar{a})$, and $p(v,\bar{b})$ in B.

Let $d(x,y,z)$ be the pseudo-Maltsev operation of U, and let

$$p'(x,\bar{y}) \stackrel{\text{def}}{=} d(p(x,\bar{y}),p(x,\bar{a}),p(v,\bar{a})).$$

With this polynomial p' we have

$$\begin{aligned} p'(u,\bar a) &= p(v,\bar a) = p'(v,\bar a), \text{ and} \\ p'(u,\bar b) &= d(p(u,\bar b),p(u,\bar a),p(v,\bar a)) \overset{\delta}{\equiv} d(p(u,\bar b),p(u,\bar b),p(v,\bar a)) \\ &= p(v,\bar a) \overset{\gamma}{\not\equiv} p(v,\bar b) = p'(v,\bar b), \end{aligned}$$

which implies (1) that we were to obtain.

Now to prove the other direction ($\Leftarrow$), suppose $\delta \geq [\beta,\theta]$, and assume that $\delta \not\geq [\theta,\beta]$. The assumption implies the failure of $C(\theta,\beta;\delta)$, and hence there exist $(u,v) \in \theta$, $(a_i,b_i) \in \beta$ for $i = 1,\cdots,n$ and $p(x,\bar y) \in \mathrm{Pol}_{n+1}\,\mathbf{A}$ such that

$$p(u,\bar a) \overset{\delta}{\equiv} p(u,\bar b) \text{ and } p(v,\bar a) \overset{\theta-\delta}{\equiv} p(v,\bar b).$$

We are going to find $p' \in \mathrm{Pol}_{n+1}\,\mathbf{A}$ such that

$$p'(u,\bar a) = p'(v,\bar a) \text{ and } p'(u,\bar b) \overset{\delta}{\not\equiv} p'(v,\bar b). \tag{2}$$

That will suffice to show $\delta \not\geq [\beta,\theta]$. In this way, the proof proceeds almost the same way as before except the place where we show that the 4 elements $p(u,\bar a)$, $p(u,\bar b)$, $p(v,\bar a)$ and $p(v,\bar b)$ all belong to the body B of a (γ,λ)-minimal set U. This time we have $p(u,\bar a) \overset{\theta}{\equiv} p(v,\bar b)$ instead of $p(u,\bar a) \overset{\beta}{\equiv} p(v,\bar b)$. But $\theta \leq \beta$ is a hypothesis of this lemma. Hence $\theta|_U \leq \beta|_U \subseteq B^2 \cup T^2$ holds as before. We skip the rest of the proof. □

Now we are ready to prove the following main theorem of this paper.

THEOREM 3. *Let β be a right nilpotent congruence in a finite algebra* $\mathbf{A}$. *Suppose that* $\mathbf{1} \notin \mathrm{typ}\,\{0_A,\beta\}$. *Then for every $k \geq 1$, we have*

$$(\beta]^k = [\beta)^k.$$

Thus the left nilpotency class of β is equal to its right nilpotency class.

Proof. We will only prove $(\beta]^k \leq [\beta)^k$. The proof of other direction $(\beta]^k \geq [\beta)^k$ is similar and hence omitted. We proceed by induction on k. For $k \leq 2$, there is nothing to prove.

Let us assume $(\beta]^k \leq [\beta)^k$ as the induction hypothesis and try to prove $(\beta]^{k+1} \leq [\beta)^{k+1}$. Let $\delta = [\beta)^{k+1}$ and $\theta = [\beta)^k$. Without loss of generality we assume that $[\beta)^k > 0_A$: i.e., $\delta < \theta$. Then by our hypothesis $\mathbf{1} \notin \operatorname{typ}\{0_A, \beta\}$ we may apply lemma 2 to get

$$[\beta)^{k+1} \geq [[\beta)^k, \beta] \Rightarrow [\beta)^{k+1} \geq [\beta, [\beta)^k].$$

Now by the induction hypothesis $[\beta)^k \geq (\beta]^k$ we have

$$[\beta)^{k+1} \geq [\beta, [\beta)^k] \geq [\beta, (\beta]^k] = (\beta]^{k+1}$$

that was to be shown. □

COROLLARY 2. *Let* β *be a right nilpotent congruence in a finite algebra* $\mathbf{A}$ *such that* $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$. *Then for every* $k \geq 1$, *we have*

$$(\beta]^k = [\beta)^k.$$

Thus the left nilpotency class of β *is equal to its right nilpotency class.*

Proof. Apply theorem 3. □

3. Discussion

The main theme of this paper is the relationship between the left and right nilpotency classes of a right nilpotent congruence β on a finite algebra $\mathbf{A}$.

If we have the commutativity of commutators, as it happens in modular varieties, it is obvious that the left and right nilpotency classes coincide.

In this paper we showed that under the condition $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$, which is strictly weaker than the condition that $\mathbf{A}$ belongs to a modular variety, the left and right nilpotency classes coincide. (By the way, under this condition $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$ it is trivial to see that K. Kearnes' result really includes the fact that left nilpotent congruences on finite algebras are also right nilpotent, as we mentioned earlier.)

For the general case when the type set of $\mathbf{A}$ may contain $\mathbf{1}$, we do not know whether the proposed upper limit for the left nilpotency class given in theorem 2 can be sharpened in any way.

References

[1] R. Freese and R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, Number 125 in Lecture Note, London Math. Soc., 1987.

[2] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Number 76 in Contemporary Mathematics., Amer. Math. Soc., 1988.

[3] Keith Kearnes, *An Order-theoretic Property of the Commutator*, Int. J. of Algebra and Computation **3** (1993), 491–533.

[4] R. McKenzie, G. McNulty, and W. Taylor, *Algebras, Lattices, Varieties Volume* 1, Wadsworth and Brooks/Cole, 1987.

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