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LINEAR OPERATORS THAT PRESERVE BOOLEAN RANKS

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ABSTRACT. We consider the Boolean linear operators that preserve Boolean rank and obtain some characterizations of the linear operators, which extend the results in [1].

1. Introduction and preliminaries

The *Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

There are many papers on linear operators that preserve the rank of matrices over several semirings. Boolean matrices also have been the subject of research by many authors. Beasley and Pullman [1] obtained characterizations of rank-preserving operators of Boolean matrices.

In this paper, we obtain new characterizations of the linear operators that preserve Boolean rank of Boolean matrices and we show that these are equivalent to those that are obtained by Beasley and Pullman in [1].

We let $\mathbb{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in the Boolean algebra $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this paper, we shall adopt the convention that $m \leq n$, unless otherwise specified.

If $\mathbb{V}$ is a nonempty subset of $\mathbb{B}^k = \mathbb{M}_{k,1}(\mathbb{B})$ containing 0 which is closed under addition then $\mathbb{V}$ is called a *Boolean vector space*. If $\mathbb{V}$ and

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$\mathbb{W}$ are vector spaces with $\mathbb{V} \subset \mathbb{W}$, then $\mathbb{V}$ is called a *subspace* of $\mathbb{W}$. We identify $\mathbb{M}_{m,n}(\mathbb{B})$ with $\mathbb{B}^{mn}$ in the usual way when we discuss it as a Boolean vector space. Let $\mathbb{V}$ be a Boolean vector space. If S is a subset of $\mathbb{V}$, then $\langle S \rangle$ denotes the intersection of all subspaces of $\mathbb{V}$ containing S, which is a subspace of $\mathbb{V}$ too, called *the subspace generated by* S. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$, then $\langle S \rangle = \left\{ \sum_{i=1}^{r} x_i \mathbf{v}_i | x_i \in \mathbb{B} \right\}$, the set of linear combinations of elements in S. Note that $\langle \phi \rangle = \{0\}$.

If an $m \times n$ Boolean matrix A is not zero, then its *Boolean rank*, $b(A)$, is the least k for which there exist $m \times k$ and $k \times n$ Boolean matrices C and D with $A = CD$. The Boolean rank of the zero matrix is 0.

We call a family of Boolean matrices consisting of 0 and some Boolean rank 1 matrices a *Boolean rank* 1 *space* if it is closed under addition.

Let $\Delta_{m,n} = \{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n\}$, and $E_{i,j}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and $\mathbb{E}_{m,n} = \{E_{i,j} | (i,j) \in \Delta_{m,n}\}$.

If $\mathbb{V}$ is a vector space over Boolean algebra $\mathbb{B}$, a mapping $T : \mathbb{V} \to \mathbb{V}$ which preserves sums and 0 is said to be a *Boolean linear operator.* A Boolean linear operator $T : \mathbb{V} \to \mathbb{V}$ is *invertible* if T is injective and $T(\mathbb{V}) = \mathbb{V}$. As with vector space over fields, the inverse, T^{-1}, of a Boolean linear operator T is also linear.

In the followings, each linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ means the Boolean linear operator.

Beasley and Pullman obtained the following:

LEMMA 1.1. [1]. *If T is a Boolean linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$, then the following statements are equivalent;*

(*a*) *T is invertible.*

(*b*) *T is injective.*

(*c*) *T permutes $\mathbb{E}_{m,n}$.*

2. Linear operators that preserve the Boolean ranks of Boolean matrices

In this section we extend the results in [1] on the linear operators that preserve Boolean rank.

Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Then,
(i) T is a (U, V)*-operator* if there exist invertible matrices U and V such that $T(A) = UAV$ for all A in $\mathbb{M}_{m,n}(\mathbb{B})$.
(ii) T is a *Boolean rank preserver* if $b(T(A)) = b(A)$ for all A in $\mathbb{M}_{m,n}(\mathbb{B})$.
(iii) T *preserves Boolean rank* k if $b(T(A)) = k$ whenever $b(A) = k$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.
(iv) T *strongly preserves Boolean rank* k provided that $b(T(A)) = k$ if and only if $b(A) = k$.

In [1], Beasley and Pullman obtained these characterizations.

THEOREM 2.1. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. The followings are equivalent:*

(a) T is invertible and preserves Boolean rank 1.
(b) T preserves Boolean ranks 1 *and* 2.
(c) T is a (U, V)-operator.
(d) T is a Boolean rank preserver.

To obtain new characterizations of the linear operators that preserve Boolean rank, we need some lemmas.

LEMMA 2.2. *If T is an invertible linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ that preserves Boolean rank k, then T strongly preserves Boolean rank k, $1 \leq k \leq m$.*

Proof. Since T preserves Boolean rank k, it suffices to show that if $b(T(A)) = k$ then $b(A) = k$. Let $S = \{A \in \mathbb{M}_{m,n}(\mathbb{B}) | b(A) = k\}$ and consider the restriction of T to S. Since T preserves Boolean rank k, $T(S) \subset S$. Thus we can write $T|_S : S \to S$. Since T is invertible, $T|_S$ is also injective and hence bijective since S is a finite set.

Let $b(T(A)) = k$ but $b(A) \neq k$ for some $A \in \mathbb{M}_{m,n}(\mathbb{B})$. Then $T(A) \in S$. Therefore we can choose $C \in S$ such that $T(C) = T|_S(C) = T(A)$. But $C \neq A$ since their Boolean ranks are different. This is a contradiction to the injectivity of T. Hence T strongly preserves Boolean rank k. □

LEMMA 2.3. *If T is an invertible linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ that preserves Boolean rank k, $1 \leq k \leq m$, then $T(\langle E_{ij}, E_{is}\rangle)$ and $T(\langle E_{ij},$*

$E_{rj}\rangle)$ *are Boolean rank* 1 *spaces for arbitrary* i *and* j *with* $1 \leq i \leq m$ *and* $1 \leq j \leq n$.

Proof. If $k = 1$, then $b(T(E_{ij} + E_{is})) = 1$ since $b(E_{ij} + E_{is}) = 1$. Thus $T(\langle E_{ij}, E_{is}\rangle)$ is a Boolean rank 1 space.

Let $k = 2$. Since T is invertible, T permutes $\mathbb{E}_{m,n}$ by Lemma 1.1. So $b(T(E_{ij} + E_{is})) = 1$ or 2. But Lemma 2.2 implies that T strongly preserves Boolean rank 2. Thus $b(T(E_{ij}+E_{is})) \neq 2$ since $b(E_{ij}+E_{is}) = 1$. Hence $T(\langle E_{ij}, E_{is}\rangle)$ is a Boolean rank 1 space.

Let $k \geq 3$. Since T permutes $\mathbb{E}_{m,n}$ by Lemma 1.1, $b(T(E_{ij} + E_{is})) = 1$ or 2. Suppose $T(\langle E_{ij}, E_{is}\rangle)$ is not a Boolean rank 1 space for some E_{ij} and E_{is}. Let $T(E_{ij}) = E_{xy}$ and $T(E_{is}) = E_{uv}$. Then $T(E_{ij} + E_{is}) = T(E_{ij}) + T(E_{is}) = E_{xy} + E_{uv}$ with $x \neq u$ and $y \neq v$. We can choose $E_{g_ih_i}$, $i = 1, 2, \cdots, k-2$, such that $b\left(\sum_{i=1}^{k-2} E_{g_ih_i}\right) = k-2$ with $g_i \neq x, u$ and $h_i \neq y, v$. Since T is invertible, we have $b\left(\sum_{i=1}^{k-2} E_{g_ih_i} + E_{xy} + E_{uv}\right) = k$ and

$$\begin{aligned} T^{-1}&\left(\sum_{i=1}^{k-2} E_{g_ih_i} + E_{xy} + E_{uv}\right) \\ &= \sum_{i=1}^{k-2} T^{-1}(E_{g_ih_i}) + T^{-1}(E_{xy}) + T^{-1}(E_{uv}) \\ &= \sum_{i=1}^{k-2} T^{-1}(E_{g_ih_i}) + E_{ij} + E_{is} \end{aligned}$$

Then $b\left(\sum_{i=1}^{k-2} T^{-1}(E_{g_ih_i}) + E_{ij} + E_{is}\right) < k$. Thus T does not strongly preserve Boolean rank k, which contradicts Lemma 2.2. Hence $T(\langle E_{ij}, E_{is}\rangle)$ is a Boolean rank 1 space.

Similarly, we can show that $T(\langle E_{ij}, E_{rj}\rangle)$ is also a Boolean rank 1 space. □

Lemma 2.4. *Let T be an invertible linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ that preserves the Boolean rank k with $1 \le k \le m$. If τ is the permutation of $\Delta_{m,n}$ representing T, then there exist permutations α, β of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$ respectively such that*

(a) $\tau(i,j) = (\alpha(i), \beta(j))$ for all $(i,j) \in \Delta_{m,n}$ or

(b) $m = n$ and $\tau(i,j) = (\beta(j), \alpha(i))$ for all $(i,j) \in \Delta_{m,n}$.

Proof. We will denote the abscissa of $\tau(i,j)$ by u_{ij} and its ordinate by v_{ij}. So $\tau(i,j) = (u_{ij}, v_{ij})$. Let $[\tau]$ be the $m \times n$ array whose (i,j)th entry is (u_{ij}, v_{ij}). Since T preserves Boolean rank k and invertible, $T(\langle E_{ij}, E_{ik}\rangle)$ and $T(\langle E_{ij}, E_{ki}\rangle)$ are Boolean rank 1 spaces by Lemma 2.3. Thus any two entries in the same row (or column) of $[\tau]$ have a common abscissa or common ordinate. It follows that if $u_{i1} = u_{i2}$ (respectively $v_{i1} = v_{i2}$) then u_{i1} is the abscissa (respectively ordinate) of each entry in the ith row of $[\tau]$. Let $\beta_i(j) = v_{ij}$ (respectively u_{ij}). Then for all i, β_i permutes $\{1, 2, \cdots, n\}$. If x were a common abscissa for one row and y were a common ordinate for another, then (x, y) would belong to both rows (because $m \le n$ and each β_i is a permutation), contradicting the injectivity of τ. Therefore either

(1) for all $(i,j) \in \Delta_{m,n}$, $u_{ij} = u_{i1}$ or

(2) for all $(i,j) \in \Delta_{m,n}$, $v_{ij} = v_{i1}$.

Suppose (1) holds. Define $\alpha(i) = u_{i1}$ for all i, $1 \le i \le m$. For some j, $v_{ij} = u_{i1}$ because β_i is a permutation. It follows that (u_{i1}, u_{i1}) occurs in the ith row of $[\tau]$ and in no other row. Thus α permutes $\{1, 2, \cdots, m\}$. Since T is invertible and preserves Boolean rank k, $T(\langle E_{1j}, E_{ij}\rangle)$ is a Boolean rank 1 space by Lemma 2.3. If $i \neq 1$, then $T(\langle E_{1j}, E_{ij}\rangle) = \langle E_{ux}, E_{vy}\rangle$ is a rank 1 space with $u = \alpha(1)$, $v = \alpha(i), x = \beta_1(j)$ and $y = \beta_i(j)$. But $\alpha(1) \neq \alpha(i)$, so $x = y$. Therefore $\beta_i = \beta_1$ for all $i \le m$. Let $\beta = \beta_1$; then $\tau(i,j) = (\alpha(i), \beta(j))$ for all $(i,j) \in \Delta_{m,n}$. If (2) holds, then $m = n$. Let $\tau'(i,j) = (v_{ij}, u_{ij})$ for all $(i,j) \in \Delta_{m,n}$, and apply (1) to τ' to complete the proof of the Lemma. □

Lemma 2.5. *If T is an invertible linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ that preserves the Boolean rank k with $1 \le k \le m$, then T is a (U,V)-operator.*

Proof. If τ is the permutation of $\Delta_{m,n}$ representing T, then, by Lemma 2.4, there exist permutations α, β of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$ respectively such that

(a) $\tau(i,j) = (\alpha(i), \beta(j))$ for all $(i,j) \in \Delta_{m,n}$ or

(b) $m = n$ and $\tau(i,j) = (\beta(j), \alpha(i))$ for all $(i,j) \in \Delta_{m,n}$.

Let π be any permutation of $\{1, 2, \cdots, k\}$. Let $E^{m,n}_{i,j}$ denote an $m \times n$ matrix of the form E_{ij}. Let $P_k(\pi) = \sum_{l=1}^{k} E^{k,k}_{l,\pi(l)}$. Then $P_k(\pi)$ is a permutation matrix. But $E^{m,n}_{i,j} E^{n,r}_{u,v} = \delta_{u,j} E^{m,r}_{i,v}$ (where $\delta_{u,j}$ is the Kronecker delta). Thus $E^{m,n}_{i,j} P_n(\pi) = E^{m,n}_{i,\pi(j)}$ and therefore $P_m(\alpha^{-1}) E^{m,n}_{i,j} P_n(\beta) = E^{m,n}_{\alpha(i),\beta(j)}$. If the case (a) holds, then we define $U = P_m(\alpha^{-1})$ and $V = P_n(\beta)$. If $A = [a_{ij}]$ is any $m \times n$ Boolean matrix, we have $A = \sum_{i,j}\{E_{i,j} | a_{ij} = 1\}$ and thus $T(A) = \sum_{i,j}\{E_{\tau(i,j)} | a_{ij} = 1\} = \sum_{i,j}\{U E_{ij} V | a_{ij} = 1\} = UAV$. If the case (b) holds, then we define $U = P_n(\beta^{-1})$ and $V = P_n(\alpha)$. Let T' be the operator on $\mathbb{M}_{m,n}(\mathbb{B})$ defined by $T'(A) = [T(A)]^t$ for all A in $\mathbb{M}_{m,n}(\mathbb{B})$. Then $T'(E_{ij}) = E_{\alpha(i),\beta(j)}$, so $T'(A) = P_n(\alpha^{-1}) A P_n(\beta)$ by the result for (a). Hence $T(A) = UA^tV$. □

THEOREM 2.6. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Then T is invertible and preserves Boolean rank k, with $1 \leq k \leq m$ if and only if T is a (U,V)-operator.*

Proof. Suppose T is a (U,V)-operator. Then T is invertible by Theorem 2.1. Let $b(A) = k$. Then $A = BC, B \in \mathbb{M}_{m,k}(\mathbb{B})$ and $C \in \mathbb{M}_{k,n}(\mathbb{B})$. Thus for $A \in \mathbb{M}_{m,n}(\mathbb{B}), T(A) = UAV = UBCV = (UB)(CV)$ where $UA \in \mathbb{M}_{m,k}(\mathbb{B})$ and $BV \in \mathbb{M}_{k,n}(\mathbb{B})$. Thus $b(T(A)) \leq k$. Suppose $b(T(A)) = l, l < k$. Then $T(A) = UAV = DE$, where $D \in \mathbb{M}_{m,l}(\mathbb{B}), E \in \mathbb{M}_{l,n}(\mathbb{B})$. That is, $A = U^{-1}DEV^{-1}$ and $b(A) \leq l$, which contradicts to $b(A) = k > l$. Thus T preserves Boolean rank k.

By Lemma 2.5, we have the converse. □

LEMMA 2.7. *If a Boolean linear operator T preserves Boolean rank 1, then we have $b(A) \geq b(T(A))$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.*

Proof. Let $b(A) = k$. Then we can write $A = A_1 + A_2 + \cdots + A_k$, where $b(A_i) = 1$ for $1 \leq i \leq k$. Thus $T(A) = T(A_1) + T(A_2) + \cdots + T(A_k)$. Since T preserves Boolean rank 1, $T(A_i)$ has Boolean rank 1. Therefore $b(T(A)) \leq k$. $\square$

THEOREM 2.8. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Then T is a (U, V)-operator if and only if T strongly preserves Boolean rank* 1.

Proof. Suppose T strongly preserves Boolean rank 1. Then T preserves Boolean rank 1. Let $b(A) = 2$. By Lemma 2.7, $b(T(A)) = 1$ or 2. If $b(T(A)) = 1$, then $b(A) = 1$ by assumption. This is a contradiction. Therefore $b(T(A)) = 2$. Thus T preserves Boolean rank 1 and 2. Then T is a (U, V)-operator by Theorem 2.1.

Now, let T be a (U, V)-operator and $b(T(A)) = 1$. Since T is a (U, V)-operator, $T(A) = UAV$ (or UA^tV if $m = n$) for some invertible matrices U and V. Then $A = U^{-1}T(A)V^{-1}$. Since $b(T(A)) = 1$, we can write $T(A) = \mathbf{a}\mathbf{b}^t$, where $\mathbf{a} \in \mathbb{M}_{m,1}(\mathbb{B}), \mathbf{b} \in \mathbb{M}_{n,1}(\mathbb{B})$. Therefore $A = U^{-1}T(A)V^{-1} = U^{-1}\mathbf{a}\mathbf{b}^tV^{-1}$ with $U^{-1}\mathbf{a} \in \mathbb{M}_{m,1}(\mathbb{B})$ and $\mathbf{b}^tV^{-1} \in \mathbb{M}_{1,n}(\mathbb{B})$. By the definition, $b(A) = 1$. Hence T strongly preserves Boolean rank 1. $\square$

COROLLARY 2.9. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Then the following are equivalent:*

(*a*) *T is invertible and preserves Boolean rank* 1.
(*b*) *T preserves Boolean ranks* 1 *and* 2.
(*c*) *T is a (U, V)-operator.*
(*d*) *T is a Boolean rank-preserver.*
(*e*) *T is invertible and preserves Boolean rank k with $k \geq 2$.*
(*f*) *T strongly preserves Boolean rank* 1.

Proof. By Theorem 2.1, we have the equivalence of (a) $\sim$ (d). Moreover Theorems 2.6 and 2.8 imply the equivalence of (c), (e) and (f). $\square$

Thus we obtain new characterizations of the Boolean linear operator that preserve the Boolean ranks of Boolean matrices, which extend those results that have been obtained in [1], as shown in the corollary 2.9.

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