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CONSTRUCTIONS FOR THE SPARSEST ORTHOGONAL MATRICES

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Abstract. In [1], it was shown that for $n \geq 2$ the least number of nonzero entries in an $n \times n$ orthogonal matrix which is not direct summable is $4n-4$, and zero patterns of the $n \times n$ orthogonal matrices with exactly $4n-4$ nonzero entries were determined. In this paper, we construct $n \times n$ orthogonal matrices with exactly $4n-4$ nonzero entries. Furthermore, we determine $m \times n$ sparse row-orthogonal matrices.

1. Introduction

An $n \times n$ matrix A is *direct summable*, if the rows and columns of A can be permuted to obtain a matrix of the form

$$\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}.$$

If A is an $n \times n$ orthogonal matrix, then it is easy to verify that if A contains a zero submatrix whose dimensions sum to n, then the submatrix complementary to it is also a zero submatrix. Hence an $n \times n$ orthogonal matrix is direct summable if and only if there exists an $r \times s$ zero submatrix of A for some positive integers r and s with $r+s=n$.

In 1991, M. Fiedler conjectured that for $n \geq 2$ an $n \times n$ orthogonal matrix which is not direct summable has at least $4n-4$ nonzero entries. In [1], this conjecture was shown in the affirmative and moreover, the zero patterns of the $n \times n$ orthogonal matrices with exactly $4n-4$

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nonzero entries were determined. B. L. Shader [5] gave a simpler proof of this result, and recently this result was extended in [2] and [3].

First, we describe a few results from [1]. Recursively define a family of $(0,1)$-matrices of order $n \geq 2$ as follows. Let

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If n is odd, define

$$\mathcal{B}_n = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & \mathcal{B}_{n-1} & & 0 \\ & & & & 1 \\ & & & & 1 \\ \hline 0 & \cdots & 0 & 1 & 1 \end{array}\right].$$

If n is even, define

$$\mathcal{B}_n = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & \mathcal{B}_{n-1} & & & \vdots \\ & & & & & 0 \\ & & & & & 1 \\ \hline 0 & \cdots & 0 & 1 & 1 & 1 \end{array}\right].$$

For example,

$$\mathcal{B}_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathcal{B}_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

As noted in [1], each of the matrices $\mathcal{B}_n$ $(n \geq 2)$ is the zero pattern of an $n \times n$ orthogonal matrix which is not direct summable and has exactly $4n-4$ nonzero entries. In addition, the matrix

$$\mathcal{R}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

is the zero pattern of a 4×4 orthogonal matrix which is not direct summable and has exactly 12 nonzero entries. Theorem 2.2 of [1] asserts that for $n \geq 2$ an $n \times n$ orthogonal matrix, Q, which is not direct summable has at least $4n-4$ nonzero entries, and if equality holds then, up to permutation of rows and columns, the zero patterns of Q are either $\mathcal{B}_n$, $\mathcal{B}_n^T$ or $\mathcal{R}_4$ for $n=4$.

In this paper, we construct $n \times n$ real orthogonal matrices with the same zero pattern as $\mathcal{B}_n$. We shall use the method by basic orthogonal matrices.

2. The construction by basic orthogonal matrices

A 2×2 orthogonal matrix $R(\theta_i)$ is a *rotation* if it has the form

$$R(\theta_i) = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \quad (\theta_i \in \mathbb{R}).$$

A 2×2 orthogonal matrix $R(\theta_i)$ is a *reflection* if it has the form

$$R(\theta_i) = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ \sin\theta_i & -\cos\theta_i \end{bmatrix}, \quad (\theta_i \in \mathbb{R}).$$

Rotations and reflections are computationally attractive because they are easily constructed by properly choosing the rotation angles or the reflection lines.

An $n \times n$ matrix Q_i is called a *basic orthogonal matrix* provided Q_i is permutation equivalent to $R(\theta_i) \oplus I_{n-2}$ for some rotation or reflection $R(\theta_i)$.

The following theorem is useful one for us.

THEOREM 2.1. *Every $n \times n$ orthogonal matrix ($n \geq 2$) can be expressed by the product of basic orthogonal matrices.*

Proof. We prove by induction on n. If $n = 2$ then this is a trivial. Assume the theorem holds for n. Suppose

$$Q = \begin{bmatrix} \widehat{Q} & \mathbf{x} \\ \mathbf{y}^T & z \end{bmatrix}$$

is an $(n+1) \times (n+1)$ orthogonal matrix where $\widehat{Q}$ is $n \times n$ real matrix. Then by singular value decomposition, $\widehat{Q}$ can be witten as

$$\widehat{Q} = U \Sigma V^T$$

where U and V are $n \times n$ orthogonal matrices, and Σ is $n \times n$ diagonal matrix with nonnegative main diagonal entries $d_1, d_2, \ldots, d_n$ and the rank of Σ is the same as the rank of $\widehat{Q}$. Thus we get

$$Q = \begin{bmatrix} U & O \\ O & I_1 \end{bmatrix} \begin{bmatrix} \Sigma & U^T\mathbf{x} \\ \mathbf{y}^T V & z \end{bmatrix} \begin{bmatrix} V^T & O \\ O & I_1 \end{bmatrix}.$$

By induction, since U and V can be expressed by the product of basic orthogonal matrices $U \oplus I_1$ and $V^T \oplus I_1$ are also respectively. It is sufficient to show that

$$Q' := \begin{bmatrix} \Sigma & U^T\mathbf{x} \\ \mathbf{y}^T V & z \end{bmatrix} = \begin{bmatrix} d_1 & & & & x_1' \\ & d_2 & & O & x_2' \\ & O & \ddots & & \vdots \\ & & & d_n & x_n' \\ y_1' & y_2' & \cdots & y_n' & z \end{bmatrix}$$

is expressed by a product of basic orthogonal matrices. Since the columns (rows) of Q' form an orthonormal set, it is easy to show that Q' is a basic orthogonal matrix. Hence an induction argument completes the proof of the theorem. □

It is useful to associate to each matrix a bipartite graph. Let $Q = [q_{ij}]$ be an $n \times n$ matrix. The *bipartite graph* of Q is the graph with vertices $1, 2, \ldots, n$ and $1', 2', \ldots, n'$ which has an edge joining i and j' if and only if $q_{ij} \neq 0$. Two vertices u and v of the graph of Q are said to be *connected* if there is a (u, v)-path in the graph of Q. Connection is an equivalence relation on the vertex set V. Thus there is a partition of V into nonempty subsets $V_1, V_2, \ldots, V_\omega$ such that two vertices u and v are connected if and only if both u and v belong to the same set V_i. The subgraphs of Q with $V_1, V_2, \ldots, V_\omega$ are called the *components* of Q. Thus if Q has exactly one component then the bipartite graph of Q is connected. It is easily verified that Q is not direct summable if and only if the bipartite graph of Q is connected.

LEMMA 2.2. *Let Q_i be an $n \times n$ basic orthogonal matrix. If $k \leq n-1$ then $Q = Q_1 Q_2 \cdots Q_k$ has at least $n-k$ components.*

Proof. We proceed by induction on k. If $k = 1$ clearly $Q = Q_1$ has at least $n - 1$ components. Let

$$Q' = Q_1 Q_2 \cdots Q_{k-1}.$$

Since Q_k is a basic orthogonal matrix there exists permutation matrices π_1 and π_2 such that $Q_k = \pi_1(R(\theta_k) \oplus I_{n-2})\pi_2$ for some rotation or reflection $R(\theta_k)$. Thus we get

$$Q = Q'Q_k = Q'\pi_1(R(\theta_k) \oplus I_{n-2})\pi_2.$$

By induction, since Q' has at least $n - k + 1$ components $Q'\pi_1$ is also. Thus we may assume that

$$Q'\pi_1 = A_1 \oplus A_2 \oplus \cdots \oplus A_{n-k+1}$$

after column and row permutation where A_i $(i = 1, 2, \ldots, n - k + 1)$ is an orthogonal matrix with suitable size. We can also assume that A_1 is orthogonal matrix which is not direct summable with the least rank among $n - k + 1$ direct summands of $Q'\pi_1$. Hence by a simple computation, if $A_1 = I_1$ then Q has at least $n-k$ components, otherwise Q has at least $n-k+1$ components. Consequently, Q has at least $n-k$ components. By induction the proof is completed. □

The following is an immediate consequence of Lemma 2.2.

COROLLARY 2.3. *Let Q_i be an $n \times n$ basic orthogonal matrix. If $Q = Q_1Q_2 \cdots Q_k$ is not direct summable then $k \geq n-1$.*

Let $\mathbb{Q}_n$ be the set of all $n\times n$ orthogonal matrices which are not direct summable with a product of exactly $n-1$ basic orthogonal matrices.

Now we are ready to construct the sparsest $n \times n$ real orthogonal matrices which are not direct summable and are expressed by a product of basic orthogonal matrices.

For an integer k with $1 \leq k \leq n-1$, define the $n\times n$ basic orthogonal matrix $\widehat{Q}_k$ by

$$\widehat{Q}_k = \begin{bmatrix} I_{k-1} & O & O \\ O & R(\theta_k) & O \\ O & O & I_{n-k-1} \end{bmatrix}$$

where $R(\theta_k)$ is a rotation or a reflection, and $0 < \theta_k < 2\pi$, $\theta_k \neq \frac{\pi}{2}, \pi, \frac{3}{2}\pi$.

Define

$$(1) \qquad \widehat{Q}_{n\times n} = \begin{cases} \widehat{Q}_1\widehat{Q}_3\cdots\widehat{Q}_{n-1}\widehat{Q}_2\widehat{Q}_4\cdots\widehat{Q}_{n-2} & \text{if } n \text{ is even,} \\ \widehat{Q}_1\widehat{Q}_3\cdots\widehat{Q}_{n-2}\widehat{Q}_2\widehat{Q}_4\cdots\widehat{Q}_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Then $\widehat{Q}_{n\times n}$ is a product of exactly $n-1$ basic orthogonal matrices, and has exactly one component from our definition of $\widehat{Q}_{n\times n}$. Thus $\widehat{Q}_{n\times n} \in \mathbb{Q}_n$.

THEOREM 2.4. *For a positive integer $n \geq 2$, let $\widehat{Q}_{n\times n}$ be the $n \times n$ matrix defined in (1). Then $\widehat{Q}_{n\times n}$ is an orthogonal matrix with exactly $4n-4$ nonzero entries which is not direct summable.*

Proof. Since $\widehat{Q}_{n\times n} \in \mathbb{Q}_n$, $\widehat{Q}_{n\times n}$ is not direct summable. If $n = 2$ then $\widehat{Q}_{n\times n}=\widehat{Q}_1 = R(\theta_k)$. Thus the theorem holds for $n = 2$. Suppose $n \geq 3$. First, for an even number $n \geq 4$ let

$$A = \widehat{Q}_1\widehat{Q}_3\cdots\widehat{Q}_{n-1}, \quad B = \widehat{Q}_2\widehat{Q}_4\cdots\widehat{Q}_{n-2}.$$

Then $\widehat{Q}_{n\times n} = AB$, and A, B are block diagonal matrices with 2×2 and 1×1 blocks, *i.e.*,

$$A = \text{diag}(\widehat{Q}_1, \widehat{Q}_3, \ldots, \widehat{Q}_{n-1}), \quad B = \text{diag}(1, \widehat{Q}_2, \widehat{Q}_4, \ldots, \widehat{Q}_{n-2}, 1).$$

Now let $\widehat{Q}_{n\times n} = [Q_{ij}]$ be a block matrix with 2×2 blocks, and let $A=[a_{ij}]$ and $B=[b_{ij}]$. Then by simple computations, we get

$$Q_{ij} = \begin{cases} \begin{bmatrix} a_{2i-1\,2i-1}b_{2i-1\,2i-1} & a_{2i-1\,2i}b_{2i\,2i} \\ a_{2i\,2i-1}b_{2i-1\,2i-1} & a_{2i\,2i}b_{2i\,2i} \end{bmatrix} \\ \quad \text{if } i=j=1,2,\dots,\frac{n}{2}, \\ \begin{bmatrix} a_{2i-1\,2i}b_{2i\,2i+1} & 0 \\ a_{2i\,2i}b_{2i\,2i+1} & 0 \end{bmatrix} \\ \quad \text{if } j=i+1 \text{ and } i=1,2,\dots,\frac{n}{2}-1, \\ \begin{bmatrix} 0 & a_{2i-1\,2i-1}b_{2i-1\,2i-2} \\ 0 & a_{2i\,2i-1}b_{2i-1\,2i-2} \end{bmatrix} \\ \quad \text{if } j=i-1 \text{ and } i=2,\dots,\frac{n}{2}, \\ O \\ \quad \text{otherwise} \end{cases}$$

where $b_{11} = b_{nn} = 1$. Thus it is easy to show that the number of nonzero entries in $\widehat{Q}_{n\times n}$ is $4n-4$. By the similar argument, we can also show that the number of nonzero entries in $\widehat{Q}_{n\times n}$ is $4n-4$ for the case of odd number n. The proof is completed. □

Note that the zero pattern of $\widehat{Q}_{n\times n}$ is precisely coincide with $\mathcal{B}_n$. So if we take a θ_k for an integer k with $1\le k\le n-1$, then we obtain a sparse orthogonal matrices with the same zero pattern as $\mathcal{B}_n$.

For example, let $n=6$. Take

$$\widehat{Q}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \widehat{Q}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\widehat{Q}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \widehat{Q}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\widehat{Q}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Then

$$\widehat{Q}_{6\times 6} = \widehat{Q}_1\widehat{Q}_3\widehat{Q}_5\widehat{Q}_2\widehat{Q}_4$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 0 & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Thus $\widehat{Q}_{6\times 6}$ is a 6×6 orthogonal matrix with exactly $4n - 4 = 20$ nonzero entries which is not direct summable, and the zero pattern of $\widehat{Q}_{6\times 6}$ is the same as $\mathcal{B}_6$.

REMARK. Let

$$X = \begin{bmatrix} \widehat{X} \\ \mathrm{x}^T \end{bmatrix}$$

be an $s \times t$ row-orthogonal matrix and let

$$Y = \begin{bmatrix} \mathrm{y}^T \\ \widehat{Y} \end{bmatrix}$$

be an $k \times l$ row-orthogonal matrix, where $\widehat{X}$ is $(s-1) \times t$ and $\widehat{Y}$ is $(k-1) \times l$. Define $X \lozenge Y$ to be the $(s+k-1) \times (t+l)$ matrix

$$X \lozenge Y = \begin{bmatrix} \widehat{X} & O \\ \mathrm{x}^T & \mathrm{y}^T \\ O & \widehat{Y} \end{bmatrix}.$$

Certainly, $X \lozenge Y$ is a row-orthogonal matrix. Since the bipartite graph of $X \lozenge Y$ is obtained from the bipartite graphs of X and Y by identifying a vertex from each, $X \lozenge Y$ is not direct summable if and only if both X and Y are not direct summable. We can extend this construction to use any number of row-orthogonal matrices by defining $X \lozenge Y \lozenge Z$ as $(X \lozenge Y) \lozenge Z$.

An $m \times n$ matrix A is *direct summable* if the rows and the columns of A can be permuted to obtain a matrix of the form

$$\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}.$$

Here, either of the matrices A_1 or A_2 may be vacuous by virtue of having no rows or no columns. But neither A_1 nor A_2 is allowed to be the 0×0 matrix. We let $\#(A)$ denote the number of nonzero entries in the matrix A.

Theorem 2.1 of [2] asserts that if A is an $m \times n$ row-orthogonal matrix which is not direct summable, then

$$\#(A) \geq \begin{cases} n + 2m - 2 & \text{if } n > 2m - 2, \\ 4m - 4 & \text{if } n \leq 2m - 2 \end{cases}. \tag{2}$$

Furthermore, equality holds in (2) if and only if for $n > 2m - 2$, the columns of A can be permuted so that

$$J \Diamond A_2 \Diamond \cdots \Diamond A_2$$

where J is the $1 \times (n - 2m + 2)$ matrix of all ones and there are $m - 1$ A_2's which are 2×2 full orthogonal matrices, and for $m < n \leq 2m - 2$, the rows and columns of A can be permuted to have the form

$$A_{k_1} \Diamond A_{k_2} \Diamond \cdots \Diamond A_{k_{n-m+1}}$$

where $k_1 + k_2 + \cdots + k_{n-m+1} = n$ ($k_i \geq 2$) and for each $i = 1, 2, \ldots, n - m + 1$, A_{k_i} is an $k_i \times k_i$ orthogonal matrix which is not direct summable with $\#(A_{k_i}) = 4k_i - 4$.

Thus we can also determine $m \times n$ sparse row-orthogonal matrices by use of sparse orthogonal matrices.

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