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FUZZY IMPLICATIVE ALGEBRAS

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ABSTRACT. We introduce the notion of a fuzzy topological implicative algebras and apply some of Foster's results [2] to homomorphic images and inverse images of fuzzy topological implicative algebras.

0. Introduction

Implicative algebras are closely related to posets with the greatest element. In [4], it concerned with some properties of implicative algebras and implicative filters. The notion of fuzzy subsets was formulated by Zadeh [6] and since then fuzzy subsets have been applied to various branches of mathematics and computer science. Rosenfeld [5] inspired the development of fuzzy algebraic structures. In [3], we discussed the fuzzification of an implicative filter in an implicative algebra. The concept of a fuzzy subset provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. Foster [2] combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by Rosenfeld [5], to formulate the elements of a theory of fuzzy topological groups. In this paper, we introduce the concept of fuzzy topological implicative algebras and apply some of Foster's results [2] to homomorphic images and inverse images of fuzzy topological implicative algebras.

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1. Preliminaries

An abstract algebra $\mathcal{X} = (X, V, \Rightarrow)$, where X is a non-empty set, V is a 0-argument operation and $\Rightarrow$ is a two-argument operation, is said to be an *implicative algebra*, provided that the following conditions are satisfied: for all $a, b, c \in X$,

(I) $a \Rightarrow a = V$,
(II) if $a \Rightarrow b = V$ and $b \Rightarrow c = V$, then $a \Rightarrow c = V$,
(III) if $a \Rightarrow b = V$ and $b \Rightarrow a = V$, then $a = b$,
(IV) $a \Rightarrow V = V$.

Let $\mathcal{X} = (X, V, \Rightarrow)$ be an implicative algebra. If we define a relation "$\leq$" as follows:

$$a \leq b \text{ if and only if } a \Rightarrow b = V,$$

then the relation "$\leq$" defines a partial order on $\mathcal{X}$. The element V is the greatest element in the poset $(A, \leq)$.

We now review some fuzzy logic concepts. Let X be a non-empty set. A fuzzy subset A in X can be characterized by a membership function $\mu_A : X \to [0,1]$. For a fuzzy subset A in X and $m \in [0,1]$, the set

$$X_A^m := \{x \in X | \mu_A(x) \geq m\}$$

is called a *level subset* of A. Let f be a mapping from a set X to a set Y. Let B be a fuzzy subset in Y with membership function μ_B. The *inverse image* of B, denoted $f^{-1}(B)$, is the fuzzy subset in X with membership function $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$. Conversely, let A be a fuzzy subset in X with membership function μ_A. Then the *image* of A, denoted by $f(A)$, is the fuzzy subset in Y such that

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x | f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy subset A in X is said to have the *sup property* if, for any subset $T \subseteq X$, there exists $t_0 \in T$ such that

$$\mu_A(t_0) = \sup_{t \in T} \mu_A(t).$$

DEFINITION 1.1. A *fuzzy topology* on a non-empty set X is a family $\mathcal{T}$ of fuzzy subsets in X which satisfying the following conditions:

(i) For all $c \in [0,1]$, $k_c \in \mathcal{T}$,

(ii) If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,

(iii) If $A_j \in \mathcal{T}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathcal{T}$,

where k_c has a constant membership function. The pair $(X, \mathcal{T})$ is called a *fuzzy topological space* and members of $\mathcal{T}$ are called *open fuzzy subsets.*

DEFINITION 1.2. Let A be a fuzzy subset in X and $\mathcal{T}$ be a fuzzy topology on X. Then the *induced fuzzy topology* on A is the family of fuzzy subsets of A which are the intersection with A of $\mathcal{T}$-open fuzzy subsets in X. The induced fuzzy topology is denoted by $\mathcal{T}_A$, and the pair $(A, \mathcal{T}_A)$ is called a *fuzzy subspace* of $(X, \mathcal{T})$.

DEFINITION 1.3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two fuzzy topological spaces. A mapping $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is said to be *fuzzy continuous* if, for each open fuzzy subset U in $\mathcal{U}$, the inverse image $f^{-1}(U)$ is in $\mathcal{T}$. Conversely, f is said to be *fuzzy open* if for each open fuzzy subset V in $\mathcal{T}$, the image $f(V)$ is in $\mathcal{U}$.

DEFINITION 1.4. Let $(A, \mathcal{T}_A)$ and $(B, \mathcal{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, respectively, and let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a mapping. Then f is also a mapping of $(A, \mathcal{T}_A)$ into $(B, \mathcal{U}_B)$ if $f(A) \subseteq B$. Furthermore f is *relatively fuzzy continuous* if for each open fuzzy set V' in $\mathcal{U}_B$, the intersection $f^{-1}(V') \cap A$ is in $\mathcal{T}_A$. Conversely, f is relatively fuzzy open if for each open fuzzy set U' in $\mathcal{T}_A$, the image $f(U')$ is in $\mathcal{U}_B$.

LEMMA 1.5 ([2]). *Let $(A, \mathcal{T}_A), (B, \mathcal{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X, \mathcal{T}), (Y, \mathcal{U})$ respectively, and let f be a fuzzy continuous mapping of $(X, \mathcal{T})$ into $(Y, \mathcal{U})$ such that $f(A) \subseteq B$. Then f is a relatively fuzzy continuous mapping of $(A, \mathcal{T}_A)$ into $(B, \mathcal{U}_B)$.*

2. Fuzzy topological implicative algebras

DEFINITION 2.1. Let $\mathcal{X} = (X, V, \Rightarrow)$ be an implicative algebra and G be a fuzzy subset in X with membership function μ_G. Then G is said to be a *fuzzy implicative algebra* in $\mathcal{X}$ if

$$\mu_G(x \Rightarrow y) - \min\{\mu_G(x), \mu_G(y)\} \geq 0$$

for all $x, y \in X$.

EXAMPLE 2.2. Let $X := \{V, a, b, c, d\}$ be a set with the following table:

$\Rightarrow$	V	a	b	c	d
V	V	a	b	c	d
a	V	V	b	c	d
b	V	a	V	c	d
c	V	V	V	V	d
d	V	V	V	V	V

Then $\mathcal{X} = (X, V, \Rightarrow)$ is an implicative algebra. Let G be a fuzzy subset in X with membership function μ_G defined by $\mu_G(V) > \mu_G(c) = \mu_G(d) > \mu_G(a) = \mu_G(b)$ is a fuzzy implicative algebra in $\mathcal{X}$.

PROPOSITION 2.3. *Let A be a fuzzy set in an implicative algebra $\mathcal{X} = (X, V, \Rightarrow)$. Then A is a fuzzy implicative algebra in $\mathcal{X}$ if and only if for every $m \in [0, 1]$, X_A^m is a subalgebra of $\mathcal{X}$, when $X_A^m \neq \emptyset$.*

Proof. Let A be a fuzzy implicative algebra in $\mathcal{X}$. Let $m \in [0, 1]$ be with $X_A^m \neq \emptyset$ If $x, y \in X_A^m$, then $\mu_A(x) \geq m$ and $\mu_A(y) \geq m$. It follows that

$$\mu_A(x \Rightarrow y) - m \geq \mu_A(x \Rightarrow y) - \min\{\mu_A(x), \mu_A(y)\} \geq 0,$$

so that $x \Rightarrow y \in X_A^m$. Hence X_A^m is a subalgebra of $\mathcal{X}$. Conversely, assume that X_A^m is not a subalgebra of $\mathcal{X}$ for some $m \in [0, 1]$. Then

$$\mu_A(x_0 \Rightarrow y_0) - \min\{\mu_A(x_0), \mu_A(y_0)\} < 0$$

for some $x_0, y_0 \in X$. If we take

$$m_0 := \frac{1}{2}\{\min\{\mu_A(x_0), \mu_A(y_0)\} - \mu_A(x_0 \Rightarrow y_0)\},$$

then clearly $m_0 \in [0,1]$, $\mu_A(x_0 \Rightarrow y_0) < m_0$ and $\min\{\mu_A(x_0), \mu_A(y_0)\} > m_0$. Consequently, $\mu_A(x_0) > m_0$ and $\mu_A(y_0) > m_0$, i.e., $x_0, y_0 \in X_A^{m_0}$. Since $X_A^{m_0}$ is a subalgebra of $\mathcal{X}$, it follows that $x_0 \Rightarrow y_0 \in X_A^{m_0}$ so that $\mu_A(x_0 \Rightarrow y_0) \geq m_0$, a contradiction. Therefore A is a fuzzy implicative algebra in $\mathcal{X}$. □

PROPOSITION 2.4. *Let f be a homomorphism of an implicative algebra $\mathcal{X} = (X, V_X, \Rightarrow_X)$ into an implicative algebra $\mathcal{Y} = (Y, V_Y, \Rightarrow_Y)$ and let G be a fuzzy implicative algebra in $\mathcal{Y}$ with membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is also a fuzzy implicative algebra in $\mathcal{X}$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu_{f^{-1}(G)}(x \Rightarrow_X y) &= \mu_G(f(x \Rightarrow_X y)) \\ &= \mu_G(f(x) \Rightarrow_Y f(y)) \\ &\geq \min\{\mu_G(f(x)), \mu_G(f(y))\} \\ &= \min\{\mu_{f^{-1}(G)}(x), \mu_{f^{-1}(G)}(y)\}, \end{aligned}$$

i.e., $\mu_{f^{-1}(G)}(x \Rightarrow_X y) - \min\{\mu_{f^{-1}(G)}(x), \mu_{f^{-1}(G)}(y)\} \geq 0$, ending the proof. □

PROPOSITION 2.5. *Let f be a homomorphism of an implicative algebra $\mathcal{X} = (X, V_X, \Rightarrow_X)$ onto an implicative algebra $\mathcal{Y} = (Y, V_Y, \Rightarrow_Y)$. If G is a fuzzy implicative algebra in $\mathcal{X}$ with the sup property, then the image $f(G)$ of G is a fuzzy implicative algebra in $\mathcal{Y}$.*

Proof. For $u, v \in Y$, let $x_0 \in f^{-1}(u), y_0 \in f^{-1}(v)$ such that

$$\mu_G(x_0) = \sup_{t \in f^{-1}(u)} \mu_G(t), \quad \mu_G(y_0) = \sup_{t \in f^{-1}(v)} \mu_G(t).$$

Then

$$\begin{aligned} \mu_{f(G)}(u \Rightarrow_Y v) &= \sup_{z \in f^{-1}(u \Rightarrow_Y v)} \mu_G(z) \\ &\geq \mu_G(x_0 \Rightarrow_X y_0) \\ &\geq \min\{\mu_G(x_0), \mu_G(y_0)\} \\ &= \min\{\sup_{t \in f^{-1}(u)} \mu_G(t), \sup_{t \in f^{-1}(v)} \mu_G(t)\} \\ &= \min\{\mu_{f(G)}(u), \mu_{f(G)}(v)\}, \end{aligned}$$

i.e., $\mu_{f(G)}(u \Rightarrow_Y v) - \min\{\mu_{f(G)}(u), \mu_{f(G)}(v)\} \geq 0$. Hence $f(G)$ is a fuzzy implicative algebra in $\mathcal{Y}$. □

Let $\mathcal{X} = (X, V, \Rightarrow)$ be an implicative algebra and let $a \in X$. We denote a_r the selfmap of X defined by $a_r(x) := x \Rightarrow a$ for all $x \in X$.

DEFINITION 2.6. Let $\mathcal{X} = (X, V, \Rightarrow)$ be an implicative algebra and $\mathcal{T}$ a fuzzy topology on X. Let G be a fuzzy implicative algebra in $\mathcal{X}$ with induced topology $\mathcal{T}_G$. Then G is said to be a *fuzzy topological implicative algebra* in $\mathcal{X}$ if for each $a \in X$ the mapping

$$a_r : x \mapsto x \Rightarrow a \text{ of } (G, \mathcal{T}_G) \rightarrow (G, \mathcal{T}_G)$$

is relatively fuzzy continuous.

THEOREM 2.7. *Given implicative algebras $\mathcal{X} = (X, V_X, \Rightarrow_X)$, $\mathcal{Y} = (Y, V_Y, \Rightarrow_Y)$ and a homomorphism $f : \mathcal{X} \rightarrow \mathcal{Y}$, let $\mathcal{U}$ and $\mathcal{T}$ be the fuzzy topologies on $\mathcal{Y}$ and $\mathcal{X}$ respectively, such that $\mathcal{T} = f^{-1}(\mathcal{U})$. If G is a fuzzy topological implicative algebra in $\mathcal{Y}$, then $f^{-1}(G)$ is a fuzzy topological implicative algebra in $\mathcal{X}$.*

Proof. We have to show that, for each $a \in X$, the mapping

$$a_r : x \mapsto x \Rightarrow_X a \text{ of } (f^{-1}(G), \mathcal{T}_{f^{-1}(G)}) \rightarrow (f^{-1}(G), \mathcal{T}_{f^{-1}(G)})$$

is relatively fuzzy continuous. Let U be an open fuzzy subset in $\mathcal{T}_{f^{-1}(G)}$ on $f^{-1}(G)$. Since f is a fuzzy continuous mapping of $(X, \mathcal{T})$ into $(Y, \mathcal{U})$, it follows from Lemma 1.5 that f is a relatively fuzzy continuous mapping of $(f^{-1}(G), \mathcal{T}_{f^{-1}(G)})$ into $(G, \mathcal{U}_G)$. Note that there exists an open fuzzy subset $V \in \mathcal{U}_G$ such that $f^{-1}(V) = U$. The membership function of $a_r^{-1}(U)$ is given by

$$\begin{aligned}&\mu_{a_r^{-1}(U)}(x) = \mu_U(a_r(x)) = \mu_U(x \Rightarrow_X a) = \mu_{f^{-1}(V)}(x \Rightarrow_X a)\\&= \mu_V(f(x \Rightarrow_X a)) = \mu_V(f(x) \Rightarrow_Y f(a)).\end{aligned}$$

Since G is a fuzzy topological implicative algebra in $\mathcal{Y}$, the mapping

$$b_r : y \mapsto y \Rightarrow_Y b \text{ of } (G, \mathcal{U}_G) \rightarrow (G, \mathcal{U}_G)$$

is relatively fuzzy continuous for each $b \in Y$. Hence

$$\begin{aligned}\mu_{a_r^{-1}(U)}(x) &= \mu_V(f(x) \Rightarrow_Y f(a)) \\ &= \mu_V(f(a)_r(f(x))) \\ &= \mu_{f(a)_r^{-1}(V)}(f(x)) \\ &= \mu_{f^{-1}(f(a)_r^{-1}(V))}(x),\end{aligned}$$

which implies that $a_r^{-1}(U) = f^{-1}(f(a)_r^{-1}(V))$ so that

$$a_r^{-1}(U) \cap f^{-1}(G) = f^{-1}(f(a)_r^{-1}(V)) \cap f^{-1}(G)$$

is open in the induced fuzzy topology on $f^{-1}(G)$. This completes the proof. □

We say that the membership function μ_G of a fuzzy implicative algebra G in an implicative algebra $\mathcal{X} = (X, V_X, \Rightarrow_X)$ is *f-invariant* [5] if, for all $x, y \in X$, $f(x) = f(y)$ implies $\mu_G(x) = \mu_G(y)$.

Clearly, a homomorphic image $f(G)$ of G is then a fuzzy implicative algebra.

THEOREM 2.8. *Given implicative algebras $\mathcal{X} = (X, V_X, \Rightarrow_X)$, $\mathcal{Y} = (Y, V_Y, \Rightarrow_Y)$ and a homomorphism f of $\mathcal{X}$ onto $\mathcal{Y}$, let $\mathcal{T}$ be the fuzzy topology on $\mathcal{X}$ and $\mathcal{U}$ be the fuzzy topology on $\mathcal{Y}$ such that $f(\mathcal{T}) = \mathcal{U}$, and let G be a fuzzy topological implicative algebra in $\mathcal{X}$. If the membership function μ_G of G is f-invariant, then $f(G)$ is a fuzzy topological implicative algebra in $\mathcal{Y}$.*

Proof. It is sufficient to show that the mapping

$$b_r : y \mapsto y \Rightarrow_Y b \text{ of } (f(G), \mathcal{U}_{f(G)}) \to (f(G), \mathcal{U}_{f(G)})$$

is relatively fuzzy continuous for each $b \in Y$. Note that f is relatively fuzzy open; for if $U' \in \mathcal{T}_G$, there exists $U \in \mathcal{T}$ such that $U' = U \cap G$ and by the f-invariance of μ_G,

$$f(U') = f(U) \cap f(G) \in \mathcal{U}_{f(G)}.$$

Let V' be an open fuzzy subset in $\mathcal{U}_{f(G)}$. Since f is onto, for each $b \in Y$ there exists $a \in X$ such that $b = f(a)$. Hence

$$\begin{aligned}\mu_{f^{-1}(b_r^{-1}(V'))}(x) &= \mu_{f^{-1}(f(a)_r^{-1}(V'))}(x)\\ &= \mu_{f(a)_r^{-1}(V')}(f(x))\\ &= \mu_{V'}(f(a)_r(f(x)))\\ &= \mu_{V'}(f(x) \Rightarrow_Y f(a))\\ &= \mu_{V'}(f(x \Rightarrow_X a))\\ &= \mu_{f^{-1}(V')}(x \Rightarrow_X a)\\ &= \mu_{f^{-1}(V')}(a_r(x))\\ &= \mu_{a_r^{-1}(f^{-1}(V'))}(x),\end{aligned}$$

which implies that $f^{-1}(b_r^{-1}(V')) = a_r^{-1}(f^{-1}(V'))$. By the hypothesis, $a_r : x \mapsto x \Rightarrow_X a$ is a relatively fuzzy continuous mapping: $(G, \mathcal{T}_G) \to (G, \mathcal{T}_G)$ and f is a relatively fuzzy continuous mapping: $(G, \mathcal{T}_G) \to (f(G), \mathcal{U}_{f(G)})$. Hence

$$f^{-1}(b_r^{-1}(V')) \cap G = a_r^{-1}(f^{-1}(V')) \cap G$$

is open in $\mathcal{T}_G$. Since f is relatively fuzzy open,

$$f(f^{-1}(b_r^{-1}(V')) \cap G) = b_r^{-1}(V') \cap f(G)$$

is open in $\mathcal{U}_{f(G)}$. This completes the proof. □

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