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FUZZY r-PREOPEN SETS AND FUZZY r-PRECONTINUOUS MAPS

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ABSTRACT. In this paper, we introduce the notions of fuzzy r-preopen (r-preclosed) sets and fuzzy r-precontinuous (r-preopen, r-preclosed) maps, and investigate some of their properties.

1. Introduction

As a generalization of sets, the concept of fuzzy sets was introduced by Zadeh. Chang [2] and Lowen [8] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Some authors [4,5,7,9] introduced new definitions of fuzzy topology as a generalization of Chang's fuzzy topology or Lowen's fuzzy topology.

Shahna [10] introduced the concepts of fuzzy preopen sets and fuzzy precontinuous maps in Chang's fuzzy topology. By generalizing these concepts, we introduce the concepts of fuzzy r-preopen sets and fuzzy r-precontinuous maps in fuzzy topological spaces. Then the concepts introduced by Shahna becomes a special case of our definitions.

2. Preliminaries

In this paper, I will denote the unit interval $[0,1]$ of the real line and $I_0 = (0,1]$. A member μ of I^X is called a fuzzy sets of X. For any $\mu \in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

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A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$.
(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.
(3) If $\mu_i \in T$ for each i, then $\bigvee \mu_i \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space.*

A *fuzzy topology* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.
(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

For each $\alpha \in (0,1]$, a *fuzzy point* x_α in X is a fuzzy set of X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x. \end{cases}$$

In this case, x and α are called the *support* and the *value* of x_α, respectively. A fuzzy point x_α is said to *belong* to a fuzzy set μ of X, denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$. A fuzzy point x_α in X is said to be *quasi-coincident* with μ, denoted by $x_\alpha \mathrm{q} \mu$, if $\alpha + \mu(x) > 1$. A fuzzy set ρ of X is said to be *quasi-coincident* with a fuzzy set μ of X, denoted by $\rho \mathrm{q} \mu$, if there is an $x \in X$ such that $\rho(x) + \mu(x) > 1$.

DEFINITION 2.1 ([6]). Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is called

(1) a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$,
(2) a *fuzzy r-closed set* of X if $\mathcal{T}(\mu^c) \geq r$.

DEFINITION 2.2 ([3]). Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\mathrm{cl}(\mu, r) = \bigwedge \{\rho \in I^X \mid \mu \leq \rho, \mathcal{T}(\rho^c) \geq r\}.$$

DEFINITION 2.3 ([6]). Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-interior* is defined by

$$\mathrm{int}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

THEOREM 2.4 ([6]). *For a fuzzy set μ of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$, we have:*

(1) $\mathrm{int}(\mu, r)^c = \mathrm{cl}(\mu^c, r)$.
(2) $\mathrm{cl}(\mu, r)^c = \mathrm{int}(\mu^c, r)$.

DEFINITION 2.5 ([6]). Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{cl}(\rho, r)$,
(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho, r) \leq \mu \leq \rho$.

DEFINITION 2.6 ([6]). Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-semiclosure* is defined by

$$\mathrm{scl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\}$$

and the *fuzzy r-semiinterior* is defined by

$$\mathrm{sint}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}.$$

DEFINITION 2.7 ([6]). Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ of X is called

(1) a *fuzzy r-neighborhood* of x_α if there is a fuzzy r-open set ρ in X such that $x_\alpha \in \rho \leq \mu$,
(2) a *fuzzy r-quasi-neighborhood* of x_α if there is a fuzzy r-open set ρ in X such that $x_\alpha q \rho \leq \mu$,
(3) a *fuzzy r-semineighborhood* of x_α if there is a fuzzy r-semiopen set ρ in X such that $x_\alpha \in \rho \leq \mu$,
(4) a *fuzzy r-quasi-semineighborhood* of x_α if there is a fuzzy r-semiopen set ρ in X such that $x_\alpha q \rho \leq \mu$.

DEFINITION 2.8 ([6]). Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-open set μ of Y,

(2) a *fuzzy r-open* map if $f(\mu)$ is a fuzzy r-open set of Y for each fuzzy r-open set μ of X,

(3) a *fuzzy r-closed* map if $f(\mu)$ is a fuzzy r-closed set of Y for each fuzzy r-closed set μ of X.

DEFINITION 2.9 ([6]). Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-open set μ of Y,

(2) a *fuzzy r-semiopen* map if $f(\mu)$ is a fuzzy r-semiopen set of Y for each fuzzy r-open set μ of X,

(3) a *fuzzy r-semiclosed* map if $f(\mu)$ is a fuzzy r-semiclosed set of Y for each fuzzy r-closed set μ of X.

3. Fuzzy r-preopen sets

DEFINITION 3.1. Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-preopen* if $\mu \le \text{int}(\text{cl}(\mu, r), r)$,

(2) *fuzzy r-preclosed* if $\text{cl}(\text{int}(\mu, r), r) \le \mu$.

THEOREM 3.2. *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent:*

(1) μ is fuzzy r-preopen.

(2) μ^c is fuzzy r-preclosed.

Proof. It follows from Theorem 2.4. □

REMARK 3.3. It is obvious that every fuzzy r-open (r-closed) set is a fuzzy r-preopen (r-preclosed) set. That the converse is false is shown by following example. It also shows that the intersection (union) of any two fuzzy r-preopen (r-preclosed) sets need not be fuzzy r-preopen (r-preclosed). Even the intersection (union) of a fuzzy r-preopen (r-preclosed) set with a fuzzy r-open (r-closed) set may fail to be fuzzy r-preopen (r-preclosed).

EXAMPLE 3.4. Let $X = \{x, y\}$ and μ_1 and μ_2 be fuzzy sets of X defined as

$$\mu_1(x) = \tfrac{1}{3}, \quad \mu_1(y) = \tfrac{2}{3};$$

and

$$\mu_2(x) = \tfrac{3}{4}, \quad \mu_2(y) = \tfrac{1}{4}.$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly, $\mathcal{T}$ is a fuzzy topology on X.

(1) Since $\text{int}(\text{cl}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \text{int}(\tilde{1}, \frac{1}{2}) = \tilde{1} \geq \mu_2$, μ_2 is fuzzy $\frac{1}{2}$-preopen. But μ_2 is not fuzzy $\frac{1}{2}$-open, because $\mathcal{T}(\mu_2) = 0$.

(2) In view of Theorem 3.2, μ_2^c is fuzzy $\frac{1}{2}$-semiclosed which is not fuzzy $\frac{1}{2}$-closed.

(3) Note μ_1 is fuzzy $\frac{1}{2}$-open and hence fuzzy $\frac{1}{2}$-preopen. Since $\text{int}(\text{cl}(\mu_1 \wedge \mu_2, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_1^c, \frac{1}{2}) = \tilde{0} \not\geq \mu_1 \wedge \mu_2$, $\mu_1 \wedge \mu_2$ is not fuzzy $\frac{1}{2}$-preopen.

(4) Clearly, μ_1^c and μ_2^c are fuzzy $\frac{1}{2}$-preclosed, but $\mu_2^c \vee \mu_3^c = (\mu_2 \wedge \mu_3)^c$ is not fuzzy $\frac{1}{2}$-preclosed.

REMARK 3.5. That fuzzy r-semiopen sets and fuzzy r-preopen sets are independent notions is shown by following example.

EXAMPLE 3.6. Let $X = \{x\}$ and μ_1, μ_2 and μ_3 be fuzzy sets of X defined as

$$\mu_1(x) = \tfrac{1}{4}, \quad \mu_2(x) = \tfrac{1}{3}, \quad \mu_3(x) = \tfrac{1}{5}.$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly, $\mathcal{T}$ is a fuzzy topology on X.

(1) Note that $\text{cl}(\text{int}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \text{cl}(\mu_1, \frac{1}{2}) = \mu_1^c \geq \mu_2$ and

$$\text{int}(\text{cl}(\mu_2, \tfrac{1}{2}), \tfrac{1}{2}) = \text{int}(\mu_1^c, \tfrac{1}{2}) = \mu_1 \not\geq \mu_2.$$

Thus μ_2 is fuzzy $\frac{1}{2}$-semiopen which is not fuzzy $\frac{1}{2}$-preopen.

(2) Note that $\text{int}(\text{cl}(\mu_3, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_1^c, \frac{1}{2}) = \mu_1 \geq \mu_3$ and

$$\text{cl}(\text{int}(\mu_3, \tfrac{1}{2}), \tfrac{1}{2}) = \text{cl}(\tilde{0}, \tfrac{1}{2}) = \tilde{0} \not\geq \mu_3.$$

Thus μ_3 is fuzzy $\frac{1}{2}$-preopen which is not fuzzy $\frac{1}{2}$-semiopen.

THEOREM 3.7. (1) *Any union of fuzzy r-preopen sets is fuzzy r-preopen.*

(2) *Any intersection of fuzzy r-preclosed sets is fuzzy r-preclosed.*

Proof. (1) Let $\{\mu_i\}$ be a collection of fuzzy r-preopen sets. Then for each i, $\mu_i \leq \text{int}(\text{cl}(\mu_i, r), r)$. Thus we have

$$\bigvee \mu_i \leq \bigvee \text{int}(\text{cl}(\mu_i, r), r) \leq \text{int}(\text{cl}(\bigvee \mu_i, r), r).$$

Hence $\bigvee \mu_i$ is a fuzzy r-preopen set.

(2) It follows from (1) using Theorem 3.2. □

DEFINITION 3.8. Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-preclosure* is defined by

$$\text{pcl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \leq \rho,\ \rho \text{ is fuzzy } r\text{-preclosed}\}$$

and the *fuzzy r-preinterior* is defined by

$$\text{pint}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \geq \rho,\ \rho \text{ is fuzzy } r\text{-preopen}\}.$$

Obviously $\text{pcl}(\mu, r)$ is the smallest fuzzy r-preclosed set which contains μ and $\text{pint}(\mu, r)$ is the greatest fuzzy r-preopen set which contained in μ. Also, $\text{pcl}(\mu, r) = \mu$ for any fuzzy r-preclosed set μ and $\text{pint}(\mu, r) = \mu$ for any fuzzy r-preopen set μ. Also we have

$$\text{int}(\mu, r) \leq \text{pint}(\mu, r) \leq \mu \leq \text{pcl}(\mu, r) \leq \text{cl}(\mu, r).$$

Moreover, we have the following results.

THEOREM 3.9. *Let $(X, \mathcal{T})$ be a fuzzy topological space and*

$$\mathrm{pcl}(\mathrm{pint}) : I^X \times I_0 \to I^X$$

the fuzzy r-preclosure (r-preinterior) operator in $(X, \mathcal{T})$. Then for $\mu, \rho \in I^X$ and $r \in I_0$,

(1) $\mathrm{pcl}(\tilde{0}, r) = \tilde{0}, \mathrm{pcl}(\tilde{1}, r) = \tilde{1}$; $\mathrm{pint}(\tilde{0}, r) = \tilde{0}, \mathrm{pint}(\tilde{1}, r) = \tilde{1}$.
(2) $\mathrm{pcl}(\mu, r) \geq \mu$; $\mathrm{pint}(\mu, r) \leq \mu$.
(3) $\mathrm{pcl}(\mu \vee \rho, r) \geq \mathrm{pcl}(\mu, r) \vee \mathrm{pcl}(\rho, r)$; $\mathrm{pint}(\mu \wedge \rho, r) \leq \mathrm{pint}(\mu, r) \wedge \mathrm{pint}(\rho, r)$.
(4) $\mathrm{pcl}(\mathrm{pcl}(\mu, r), r) = \mathrm{pcl}(\mu, r)$; $\mathrm{pint}(\mathrm{pint}(\mu, r), r) = \mathrm{pint}(\mu, r)$.

Proof. It is obvious. □

THEOREM 3.10. *For a fuzzy set μ of a fuzzy topological space X and $r \in I_0$,*

(1) $\mathrm{pint}(\mu, r)^c = \mathrm{pcl}(\mu^c, r)$.
(2) $\mathrm{pcl}(\mu, r)^c = \mathrm{pint}(\mu^c, r)$.

Proof. (1) Since $\mathrm{pint}(\mu, r) \leq \mu$ and $\mathrm{pint}(\mu, r)$ is fuzzy r-preopen in X, $\mu^c \leq \mathrm{pint}(\mu, r)^c$ and $\mathrm{pint}(\mu, r)^c$ is fuzzy r-preclosed in X. Thus

$$\mathrm{pcl}(\mu^c, r) \leq \mathrm{pcl}(\mathrm{pint}(\mu, r)^c, r) = \mathrm{pint}(\mu, r)^c.$$

Conversely, since $\mu^c \leq \mathrm{pcl}(\mu^c, r)$ and $\mathrm{pcl}(\mu^c, r)$ is fuzzy r-preclosed in X, $\mathrm{pcl}(\mu^c, r)^c \leq \mu$ and $\mathrm{pcl}(\mu^c, r)^c$ is fuzzy r-preopen in X. Thus

$$\mathrm{pcl}(\mu^c, r)^c = \mathrm{pint}(\mathrm{pcl}(\mu^c, r)^c, r) \leq \mathrm{pint}(\mu, r)$$

and hence $\mathrm{pint}(\mu, r)^c \leq \mathrm{pcl}(\mu^c, r)$.
(2) Similar to (1). □

THEOREM 3.11. *For a fuzzy set μ of a fuzzy topological space X and $r \in I_0$,*

(1) $\mathrm{pint}(\mathrm{pcl}(\mathrm{pint}(\mathrm{pcl}(\mu, r), r), r), r) = \mathrm{pint}(\mathrm{pcl}(\mu, r), r)$.
(2) $\mathrm{pcl}(\mathrm{pint}(\mathrm{pcl}(\mathrm{pint}(\mu, r), r), r), r) = \mathrm{pcl}(\mathrm{pint}(\mu, r), r)$.

Proof. (1) Since $\operatorname{pint}(\operatorname{pcl}(\mu,r),r)$ is fuzzy r-preopen and

$$\operatorname{pint}(\operatorname{pcl}(\mu,r),r) \leq \operatorname{pcl}(\operatorname{pint}(\operatorname{pcl}(\mu,r),r),r),$$

it follows that

$$\begin{aligned}\operatorname{pint}(\operatorname{pcl}(\mu,r),r) &= \operatorname{pint}(\operatorname{pint}(\operatorname{pcl}(\mu,r),r),r)\\ &\leq \operatorname{pint}(\operatorname{pcl}(\operatorname{pint}(\operatorname{pcl}(\mu,r),r),r),r).\end{aligned}$$

Conversely, since $\operatorname{pint}(\operatorname{pcl}(\mu,r),r) \leq \operatorname{pcl}(\mu,r)$ and $\operatorname{pcl}(\mu,r)$ is fuzzy r-preclosed in X, it follows that

$$\operatorname{pcl}(\operatorname{pint}(\operatorname{pcl}(\mu,r),r),r) \leq \operatorname{pcl}(\operatorname{pcl}(\mu,r),r) = \operatorname{pcl}(\mu,r).$$

Thus $\operatorname{pint}(\operatorname{pcl}(\operatorname{pint}(\operatorname{pcl}(\mu,r),r),r),r) \leq \operatorname{pint}(\operatorname{pcl}(\mu,r),r)$.

(2) Similar to (1). □

Let $(X,\mathcal{T})$ be a fuzzy topological space. For an r-cut $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$, it is obvious that $(X,\mathcal{T}_r)$ is a Chang's fuzzy topological space for all $r \in I_0$.

Let (X,T) be a Chang's fuzzy topological space and $r \in I_0$. Recall [4] that a fuzzy topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

The next two theorems show that a fuzzy preopen set is a special case of a fuzzy r-preopen set.

THEOREM 3.12. *Let μ be a fuzzy set of a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then μ is fuzzy r-preopen (r-preclosed) in $(X,\mathcal{T})$ if and only if μ is fuzzy preopen (preclosed) set in $(X,\mathcal{T}_r)$.*

Proof. Straightforward. □

THEOREM 3.13. *Let μ be a fuzzy set of a Chang's fuzzy topological space (X,T) and $r \in I_0$. Then μ is fuzzy preopen (preclosed) in (X,T) if and only if μ is fuzzy r-preopen (r-preclosed) in (X,T^r).*

Proof. Straightforward. □

4. Fuzzy r-preneighborhoods

Now, we are going to introduce fuzzy r-preneighborhoods and fuzzy r-quasi-preneighborhoods.

DEFINITION 4.1. Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ of X is called

(1) a *fuzzy r-preneighborhood* of x_α if there is a fuzzy r-preopen set ρ in X such that $x_\alpha \in \rho \le \mu$,
(2) a *fuzzy r-quasi-preneighborhood* of x_α if there is a fuzzy r-preopen set ρ in X such that $x_\alpha \mathrm{q} \rho \le \mu$.

THEOREM 4.2. *Let $(X, \mathcal{T})$ be a fuzzy topological space and $r \in I_0$. Then*

(1) *a fuzzy set μ of X is fuzzy r-preopen if and only if μ is a fuzzy r-preneighborhood of x_α for every fuzzy point $x_\alpha \in \mu$,*
(2) *a fuzzy set μ of X is fuzzy r-preopen if and only if μ is a fuzzy r-quasi-preneighborhood of x_α for every fuzzy point $x_\alpha \mathrm{q} \mu$.*

Proof. (1) Let μ be fuzzy r-preopen of X and $x_\alpha \in \mu$. Put $\rho = \mu$. Then ρ is fuzzy r-preopen of X and $x_\alpha \in \rho \le \mu$. Thus μ is a fuzzy r-preneighborhood of x_α.

Conversely, let $x_\alpha \in \mu$. Since μ is a fuzzy r-preneighborhood of x_α, there is a fuzzy r-preopen set ρ_{x_α} in X such that $x_\alpha \in \rho_{x_\alpha} \le \mu$. So we have

$$\mu = \bigvee\{x_\alpha \mid x_\alpha \in \mu\} \le \bigvee\{\rho_{x_\alpha} \mid x_\alpha \in \mu\} \le \mu$$

and hence $\mu = \bigvee\{\rho_{x_\alpha} \mid x_\alpha \in \mu\}$. Since each ρ_{x_α} is fuzzy r-preopen, μ is fuzzy r-preopen.

(2) Let μ be fuzzy r-preopen of X and $x_\alpha \mathrm{q} \mu$. Put $\rho = \mu$. Then ρ is fuzzy r-preopen of X and $x_\alpha \mathrm{q} \rho \le \mu$. Thus μ is a fuzzy r-quasi-preneighborhood of x_α.

Conversely, let x_α be any fuzzy point in μ such that $\alpha < \mu(x)$. Then $x_{1-\alpha} \mathrm{q} \mu$. By hypothesis, μ is a fuzzy r-quasi-preneighborhood of $x_{1-\alpha}$. Thus there is a fuzzy r-preopen set ρ_{x_α} in X such that $x_{1-\alpha} \mathrm{q} \rho_{x_\alpha} \le \mu$.

Hence $\alpha < \rho_{x_\alpha}(x)$ and $\rho_{x_\alpha} \leq \mu$. So we have

$$\begin{aligned}\mu &= \bigvee\{x_\alpha \mid x_\alpha \text{ is a fuzzy point in } \mu \text{ such that } \alpha < \mu(x)\}\\ &\leq \bigvee\{\rho_{x_\alpha} \mid x_\alpha \text{ is a fuzzy point in } \mu \text{ such that } \alpha < \mu(x)\}\\ &\leq \mu\end{aligned}$$

and hence $\mu = \bigvee\{\rho_{x_\alpha} \mid x_\alpha$ is a fuzzy point in μ such that $\alpha < \mu(x)\}$. Since each ρ_{x_α} is fuzzy r-preopen, μ is fuzzy r-preopen. □

THEOREM 4.3. *Let x_α be a fuzzy point in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then $x_\alpha \in \text{pcl}(\mu, r)$ if and only if $\rho q \mu$ for all fuzzy r-quasi-preneighborhood ρ of x_α,*

Proof. Suppose that there is a fuzzy r-quasi-preneighborhood ρ of x_α such that $\rho \not{q} \mu$. Then there is a fuzzy r-preopen set λ such that $x_\alpha q \lambda \leq \rho$. So $\lambda \not{q} \mu$ and hence $\mu \leq \lambda^c$. Since λ^c is fuzzy r-preclosed, $\text{pcl}(\mu, r) \leq \text{pcl}(\lambda^c, r) = \lambda^c$. On the other hand, since $x_\alpha q \lambda$, $x_\alpha \notin \lambda^c$. Hence $x_\alpha \notin \text{pcl}(\mu, r)$. It is a contradiction.

Conversely, suppose $x_\alpha \notin \text{pcl}(\mu, r)$. Then there is a fuzzy r-preclosed set η such that $\mu \leq \eta$ and $x_\alpha \notin \eta$. Thus η^c is fuzzy r-preopen and $x_\alpha q \eta^c$, and hence η^c is a fuzzy r-quasi-preneighborhood of x_α. By hypothesis, $\eta^c q \mu$ and hence $\mu \not\leq (\eta^c)^c = \eta$. It is a contradiction. □

Clearly, every fuzzy r-neighborhood (r-quasi-neighborhood) of x_α is also a fuzzy r-preneighborhood (r-quasi-preneighborhood) of x_α. The converse does not hold as in the following example.

EXAMPLE 4.4. Let $X = \{x\}$ and μ_1 and μ_2 be fuzzy sets of X defined as

$$\mu_1(x) = \tfrac{1}{4}, \quad \mu_2(x) = \tfrac{4}{5}.$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly, $\mathcal{T}$ is a fuzzy topology on X. Let $\alpha = \frac{1}{3}$. Then μ_2 is a fuzzy $\frac{1}{2}$-preneighborhood of x_α which is not a fuzzy $\frac{1}{2}$-neighborhood of x_α. Also μ_2 is a fuzzy $\frac{1}{2}$-quasi-preneighborhood of x_α which is not a fuzzy $\frac{1}{2}$-quasi-neighborhood of x_α.

The next two theorems show the relation between a fuzzy preneighborhood and a fuzzy r-preneighborhood.

THEOREM 4.5. *Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ is a fuzzy r-preneighborhood (r-quasi-preneighborhood) of x_α in $(X, \mathcal{T})$ if and only if μ is a fuzzy preneighborhood (quasi-preneighborhood) of x_α in $(X, \mathcal{T}_r)$.*

Proof. Straightforward. □

THEOREM 4.6. *Let x_α be a fuzzy point of a Chang's fuzzy topological space (X, T) and $r \in I_0$. Then a fuzzy set μ is a fuzzy preneighborhood (quasi- preneighborhood) of x_α in (X, T) if and only if μ is a fuzzy r-preneighborhood (r-quasi-preneighborhood) of x_α in (X, T^r).*

Proof. Straightforward. □

5. Fuzzy r-precontinuous maps

DEFINITION 5.1. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-precontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-preopen set of X for each fuzzy r-open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-preclosed set of X for each fuzzy r-closed set μ of Y,

(2) a *fuzzy r-preopen* map if $f(\rho)$ is a fuzzy r-preopen set of Y for each fuzzy r-open set ρ of X,

(3) a *fuzzy r-preclosed* map if $f(\rho)$ is a fuzzy r-preclosed set of Y for each fuzzy r-closed set ρ of X.

REMARK 5.2. It is obvious that every fuzzy r-continuous (r-open, r-closed) map is also a fuzzy r-precontinuous (r-preopen, r-preclosed) map. That the converse is false is shown by the following example.

EXAMPLE 5.3. Let $X = \{x\}$ and μ_1 and μ_2 be fuzzy sets of X defined as

$$\mu_1(x) = \tfrac{1}{3}, \quad \mu_2(x) = \tfrac{1}{4}.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly, $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on X.

(1) Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_2) = \mu_2$ are fuzzy $\frac{1}{2}$-preopen sets of $(X, \mathcal{T}_1)$ and hence f is fuzzy $\frac{1}{2}$-precontinuous. On the other hand, $f^{-1}(\mu_2) = \mu_2$ is not fuzzy $\frac{1}{2}$-open in $(X, \mathcal{T}_1)$ and hence f is not fuzzy $\frac{1}{2}$-continuous.

(2) Consider the map $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ defined by $f(x) = x$. Then $f(\tilde{0}) = \tilde{0}$, $f(\tilde{1}) = \tilde{1}$ and $f(\mu_2) = \mu_2$ are fuzzy $\frac{1}{2}$-preopen sets of $(X, \mathcal{T}_1)$ and hence f is fuzzy $\frac{1}{2}$-preopen. On the other hand, $f(\mu_2) = \mu_2$ is not fuzzy $\frac{1}{2}$-open in $(X, \mathcal{T}_1)$ and hence f is not fuzzy $\frac{1}{2}$-open.

(3) Consider the map $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ defined by $f(x) = x$. Then $f(\tilde{0}) = \tilde{0}$, $f(\tilde{1}) = \tilde{1}$ and $f(\mu_2^c) = \mu_2^c$ are fuzzy $\frac{1}{2}$-preclosed sets of $(X, \mathcal{T}_1)$ and hence f is fuzzy $\frac{1}{2}$-preclosed. On the other hand, $f(\mu_2^c) = \mu_2^c$ is not fuzzy $\frac{1}{2}$-closed in $(X, \mathcal{T}_1)$ and hence f is not fuzzy $\frac{1}{2}$-closed.

Now, we characterize fuzzy r-precontinuity by fuzzy r-closure and fuzzy r-interior.

THEOREM 5.4. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-precontinuous map.*

(2) *$\mathrm{cl}(\mathrm{int}(f^{-1}(\mu), r), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ for each fuzzy set μ of Y.*

(3) *$f(\mathrm{cl}(\mathrm{int}(\rho, r), r)) \leq \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*

Proof. (1) $\Rightarrow$ (2) Let f be a fuzzy r-precontinuous map and μ a fuzzy set of Y. Then $\mathrm{cl}(\mu, r)$ is a fuzzy r-closed set of Y. Since f is fuzzy r-precontinuous, $f^{-1}(\mathrm{cl}(\mu, r))$ is a fuzzy r-preclosed set of X. Thus

$$f^{-1}(\mathrm{cl}(\mu, r)) \geq \mathrm{cl}(\mathrm{int}(f^{-1}(\mathrm{cl}(\mu, r)), r), r) \geq \mathrm{cl}(\mathrm{int}(f^{-1}(\mu), r), r).$$

(2) $\Rightarrow$ (3) Let ρ be a fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. By (2),

$$f^{-1}(\mathrm{cl}(f(\rho), r)) \geq \mathrm{cl}(\mathrm{int}(f^{-1}f(\rho), r), r) \geq \mathrm{cl}(\mathrm{int}(\rho, r), r).$$

Hence

$$\mathrm{cl}(f(\rho), r) \geq ff^{-1}(\mathrm{cl}(f(\rho), r)) \geq f(\mathrm{cl}(\mathrm{int}(\rho, r), r)).$$

(3) $\Rightarrow$ (1) Let μ be a fuzzy r-closed set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (3),

$$f(\mathrm{cl}(\mathrm{int}(f^{-1}(\mu), r), r)) \leq \mathrm{cl}(ff^{-1}(\mu), r) \leq \mathrm{cl}(\mu, r) = \mu.$$

So

$$\mathrm{cl}(\mathrm{int}(f^{-1}(\mu), r), r) \leq f^{-1}f(\mathrm{cl}(\mathrm{int}(f^{-1}(\mu), r), r)) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu)$ is a fuzzy r-preclosed set of X and hence f is fuzzy r-precontinuous. □

The definition of fuzzy r-precontinuity can be restated in terms of fuzzy r-preclosure and fuzzy r-preinterior.

THEOREM 5.5. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-precontinuous map.*
(2) *$f(\mathrm{pcl}(\rho, r)) \leq \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.*
(3) *$\mathrm{pcl}(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ for each fuzzy set μ of Y.*
(4) *$f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{pint}(f^{-1}(\mu), r)$ for each fuzzy set μ of Y.*

Proof. (1) $\Rightarrow$ (2) Let ρ be a fuzzy set of X. Since $\mathrm{cl}(f(\rho),r)$ is a fuzzy r-closed set of Y, $f^{-1}(\mathrm{cl}(f(\rho),r))$ is a fuzzy r-preclosed set of X. Thus

$$\begin{aligned}\mathrm{pcl}(\rho,r) &\le \mathrm{pcl}(f^{-1}f(\rho),r)\\ &\le \mathrm{pcl}(f^{-1}(\mathrm{cl}(f(\rho),r)),r) = f^{-1}(\mathrm{cl}(f(\rho),r)).\end{aligned}$$

Hence we have

$$f(\mathrm{pcl}(\rho,r)) \le ff^{-1}(\mathrm{cl}(f(\rho),r)) \le \mathrm{cl}(f(\rho),r).$$

(2) $\Rightarrow$ (3) Let μ be a fuzzy set of Y. By (2),

$$f(\mathrm{pcl}(f^{-1}(\mu),r)) \le \mathrm{cl}(ff^{-1}(\mu),r) \le \mathrm{cl}(\mu,r).$$

Thus we have

$$\mathrm{pcl}(f^{-1}(\mu),r) \le f^{-1}f(\mathrm{pcl}(f^{-1}(\mu),r)) \le f^{-1}(\mathrm{cl}(\mu,r)).$$

(3) $\Rightarrow$ (4) Let μ be a fuzzy set of Y. Then μ^c is a fuzzy set of Y. By (3),

$$\mathrm{pcl}(f^{-1}(\mu)^c,r) = \mathrm{pcl}(f^{-1}(\mu^c),r) \le f^{-1}(\mathrm{cl}(\mu^c,r)).$$

Thus we have

$$f^{-1}(\mathrm{int}(\mu,r)) = f^{-1}(\mathrm{cl}(\mu^c,r)^c) \le \mathrm{pcl}(f^{-1}(\mu)^c,r)^c = \mathrm{pint}(f^{-1}(\mu),r).$$

(4) $\Rightarrow$ (1) Let μ be a fuzzy r-open set of Y. Then $\mathrm{int}(\mu,r)=\mu$. By (4),

$$f^{-1}(\mu) = f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{pint}(f^{-1}(\mu),r) \le f^{-1}(\mu).$$

Thus $f^{-1}(\mu) = \mathrm{pint}(f^{-1}(\mu),r)$. Hence $f^{-1}(\mu)$ is a fuzzy r-preopen set of X. Therefore f is fuzzy r-precontinuous. □

THEOREM 5.6. *Let $f:(X,\mathcal{T})\to(Y,\mathcal{U})$ be a bijection and $r\in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-precontinuous map.*
(2) *$f(\mathrm{pcl}(\rho,r)) \le \mathrm{cl}(f(\rho),r)$ for each fuzzy set ρ of X.*
(3) *$\mathrm{pcl}(f^{-1}(\mu),r) \le f^{-1}(\mathrm{cl}(\mu,r))$ for each fuzzy set μ of Y.*
(4) *$f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{pint}(f^{-1}(\mu),r)$ for each fuzzy set μ of Y.*
(5) *$\mathrm{int}(f(\rho),r) \le f(\mathrm{pint}(\rho,r))$ for each fuzzy set ρ of X.*

Proof. By Theorem 5.5, it suffices to show that (4) is equivalent to (5). Let ρ be any fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. Since f is one-to-one,

$$f^{-1}(\text{int}(f(\rho),r) \leq \text{pint}(f^{-1}f(\rho),r) = \text{pint}(\rho,r).$$

Since f is onto,

$$\text{int}(f(\rho),r) = ff^{-1}(\text{int}(f(\rho),r)) \leq f(\text{pint}(\rho,r)).$$

Conversely, let μ be any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. Since f is onto,

$$\text{int}(\mu,r) = \text{int}(ff^{-1}(\mu),r) \leq f(\text{pint}(f^{-1}(\mu),r)).$$

Since f is one-to-one,

$$f^{-1}(\text{int}(\mu,r)) \leq f^{-1}f(\text{pint}(f^{-1}(\mu),r) = \text{pint}(f^{-1}(\mu),r).$$

Hence the theorem follows. □

THEOREM 5.7. *Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-preopen map.*
(2) *$f(\text{int}(\rho,r)) \leq \text{pint}(f(\rho),r)$ for each fuzzy set ρ of X.*
(3) *$\text{int}(f^{-1}(\mu),r) \leq f^{-1}(\text{pint}(\mu,r))$ for each fuzzy set μ of Y.*

Proof. (1) ⇒ (2) Let ρ be a fuzzy set of X. Clearly $\text{int}(\rho,r)$ is a fuzzy r-open set of X. Since f is a fuzzy r-preopen map, $f(\text{int}(\rho,r))$ is a fuzzy r-preopen set of Y. Also since $f(\text{int}(\rho,r)) \leq f(\rho)$,

$$f(\text{int}(\rho,r)) = \text{pint}(f(\text{int}(\rho,r)),r) \leq \text{pint}(f(\rho),r).$$

(2) ⇒ (3) Let μ be a fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (2),

$$f(\text{int}(f^{-1}(\mu),r)) \leq \text{pint}(ff^{-1}(\mu),r) \leq \text{pint}(\mu,r).$$

Thus we have

$$\operatorname{int}(f^{-1}(\mu), r) \le f^{-1}f(\operatorname{int}(f^{-1}(\mu), r)) \le f^{-1}(\operatorname{pint}(\mu, r)).$$

(3) $\Rightarrow$ (1) Let ρ be a fuzzy r-open set of X. Then $\operatorname{int}(\rho, r) = \rho$ and $f(\rho)$ is a fuzzy set of Y. By (3),

$$\rho = \operatorname{int}(\rho, r) \le \operatorname{int}(f^{-1}f(\rho), r) \le f^{-1}(\operatorname{pint}(f(\rho), r)).$$

So we have

$$f(\rho) \le ff^{-1}(\operatorname{pint}(f(\rho), r)) \le \operatorname{pint}(f(\rho), r) \le f(\rho).$$

Thus $f(\rho) = \operatorname{pint}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy r-preopen set of Y. Therefore f is fuzzy r-precontinuous. □

THEOREM 5.8. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-preclosed map.*
(2) *$\operatorname{pcl}(f(\rho), r) \le f(\operatorname{cl}(\rho, r))$ for each fuzzy set ρ of X.*

Proof. (1) $\Rightarrow$ (2) Let ρ be a fuzzy set of X. Clearly $\operatorname{cl}(\rho, r)$ is a fuzzy r-closed set of X. Since f is a fuzzy r-preclosed map, $f(\operatorname{cl}(\rho, r))$ is a fuzzy r-preclosed set of Y. Since $f(\rho) \le f(\operatorname{cl}(\rho, r))$,

$$\operatorname{pcl}(f(\rho), r) \le \operatorname{pcl}(f(\operatorname{cl}(\rho, r)), r) = f(\operatorname{cl}(\rho, r)).$$

(2) $\Rightarrow$ (1) Let ρ be a fuzzy r-closed set of X. Then $\operatorname{cl}(\rho, r) = \rho$. By (2),

$$\operatorname{pcl}(f(\rho), r) \le f(\operatorname{cl}(\rho, r)) = f(\rho) \le \operatorname{pcl}(f(\rho), r).$$

Thus $f(\rho) = \operatorname{pcl}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy r-preclosed set of Y. Therefore f is fuzzy r-preclosed. □

THEOREM 5.9. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-preclosed map.*
(2) *$\operatorname{pcl}(f(\rho), r) \le f(\operatorname{cl}(\rho, r))$ for each fuzzy set ρ of X.*
(3) *$f^{-1}(\operatorname{pcl}(\mu, r)) \le \operatorname{cl}(f^{-1}(\mu), r)$ for each fuzzy set μ of Y.*

Proof. By Theorem 5.8, it suffices to show that (2) is equivalent to (3). Let μ be any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. Since f is onto,

$$\mathrm{pcl}(\mu, r) = \mathrm{pcl}(ff^{-1}(\mu), r) \leq f(\mathrm{cl}(f^{-1}(\mu), r).$$

Since f is one-to-one,

$$f^{-1}(\mathrm{pcl}(\mu, r)) \leq f^{-1}f(\mathrm{cl}(f^{-1}(\mu), r)) = \mathrm{cl}(f^{-1}(\mu), r).$$

Conversely, let ρ be any fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. Since f is one-to-one,

$$f^{-1}(\mathrm{pcl}(f(\rho), r)) \leq \mathrm{cl}(f^{-1}f(\rho), r) = \mathrm{cl}(\rho, r).$$

Since f is onto,

$$\mathrm{pcl}(f(\rho), r) = ff^{-1}(\mathrm{pcl}(f(\rho), r)) \leq f(\mathrm{cl}(\rho, r)).$$

Hence the theorem follows. □

The next two theorems show that a fuzzy precontinuous map is a special case of a fuzzy r-precontinuous map.

THEOREM 5.10. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is fuzzy r-precontinuous (r-preopen and r-preclosed, respectively) if and only if $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$ is fuzzy precontinuous (preopen and preclosed, respectively).*

Proof. Straightforward. □

THEOREM 5.11. *Let $f : (X, T) \to (Y, U)$ be a map from a Chang's fuzzy topological space X to another Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy precontinuous (preopen and preclosed, respectively) if and only if $f : (X, T^r) \to (Y, U^r)$ is fuzzy r-precontinuous (r-preopen and r-preclosed, respectively).*

Proof. Straightforward. □

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