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PRODUCTS OF MANIFOLDS AS CODIMENSION k FIBRATORS

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ABSTRACT. In this paper, we show that any product of a closed orientable n-manifold N_1 with finite fundamental group and a closed orientable asherical m-manifold N_2 with hopfian fundamental group, where $\chi(N_1)$ and $\chi(N_2)$ are nonzero, is a codimension 2 fibrator. Moreover, if $\pi_i(N_1) = 0$ for $1 < i < k$, then $N_1 \times N_2$ is a codimension k PL fibrator.

1. Introduction

In studying proper maps between manifolds, approximate fibrations introduced and studied by Coram and Duvall [2] form an important class of mappings nearly as effective as Hurewicz fibrations.

A proper map $p : M \to B$ between locally compact ANRs is called an approximate fibration if it has the following homotopy property: Given an open cover ϵ of B, an arbitrary space X and two maps $g : X \to M$ and $F : X \times I \to B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \to M$ such that $G_0 = g$ and $p \circ G$ is ϵ-close to F.

When $p : M \to B$ is an approximate fibration, there is a homotopy exact sequence developed by Coram and Duvall [2];

$$\cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}b) \to \pi_i(M) \to \pi_i(B) \to \cdots$$

just like the one for Hurewicz fibrations, relating homotopy data of the total space, base space, and typical fiber.

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An extensive variety of closed n-manifolds N, called codimension k (orientable, respectively) fibrators, automatically induce approximate fibrations, in the sense that all proper maps $f: M \to B$ from any (orientable, respectively) $(n+k)$-manifold M to a k-dimensional ANR B such that each $f^{-1}(b)$ has the same homotopy type (or, more generally, the same shape) as N are approximate fibrations.

The main problem is to determine which manifolds N are codimension k (orientable, respectively) fibrators.

Most closed manifolds are known to be codimension 1 orientable fibrators [4]. For codimension 2 (orientable) fibrators, we have fairly rich data [1,5,6,7,11,15,16]. In particular, every closed surface except torus and projective plane is a codimension 2 fibrator [5]. Also manifolds that satisfy a certain hopfian property are codimension 2 orientable fibrators if they have either non-zero Euler characteristic or hyperhopfian fundamental groups [7]. Recently, Im and Kim [17] extended so that they are also codimension 2 fibrators.

In section 3, we restrict objects to the PL category and all manifolds are orientable. Restriction to the PL category offers some advantages. The target spaces are standard geometric objects, obviously finite dimensional and locally contractible, features which a priori dispel potentially troublesome issues lurking in the background of the general (non-PL) setting [5]. The chief benefit is not the simplicial structure of the image, however, but rather the potential for inductive arguments, as in the classical PL topology, which apply to the restriction of p over certain links in the target and bring about the lowering of fiber codimension without changing fiber character. We call N a codimension k PL fibrator if, for all $(n+k)$-manifolds M and N-like PL maps $p: M \to B$(polyhedron), p is an approximate fibration. If N has this property for all $k > 0$, call N simply a PL fibrator.

Surprisingly many manifolds are known to be codimension k PL fibrators [8,9]. If N is a closed, aspherical manifold which is a codimension 2 PL fibrator, then N is a codimension 3 PL fibrator. Moreover, if N^n is a closed aspherical manifold with certain fundamental group, then N^n is a PL fibrator.

So far, the matter of closure with respect to cartesian products of codimension k fibrators is still open. On this front, Im [15,16] has shown that all cartesian products of surfaces of non-negative Euler characteristic are codimension 2 fibrators. To determine whether any product of finitely many codimension-2 fibrators is a codomension-2 fibrator, one may confront the question, which is widely open, whether the collection of hopfian manifolds is closed under the product operator.

In this paper, we show that any product of a closed orientable n-manifold N_1 with finite fundamental group and a closed orientable asherical m-manifold N_2 with hopfian fundamental group, where $\chi(N_1)$ and $\chi(N_2)$ are nonzero, is a codimension 2 fibrator. Moreover, if $\pi_i(N_1) = 0$ for $1 < i < k$, then $N_1 \times N_2$ is a codimension k PL fibrator.

A group G is hopfian if every epimorphism $\Theta : G \to G$ is necessarily an isomorphism, while a finitely presented group G is hyperhopfian if every homomorphism $\Psi : G \to G$ with $\Psi(G)$ normal and $G/\Psi(G)$ cyclic is an automorphism. A group G is normally cohopfian if every monomorphism $\Phi : G \to G$ with normal image is an automorphism, and a group G is cohopfian if every monomorphism $\Phi : G \to G$ is an automorphism.

A closed manifold N is hopfian if it is orientable and every degree one map $h : N \to N$ which induces a π_1-isomorphism is a homotopy equivalence. This term plays an important role in determining approximate fibrations. Swarup [20] has shown this hopfian feature for closed orientable n-manifolds N with $\pi_i(N) = 0$ for $1 < i < n-1$, and Hausmann has done the same for all closed orientable 4-manifolds and all closed orientable manifolds with nilpotent fundamental group [13].

The (absolute) degree of a map $f : N \to N$, where N is a closed, connected, orientable n-manifold, is the non-negative integer such that the induced endomorphism of $H_n(N : Z) \cong Z$ amounts to multiplication by d, up to sign.

Homology is computed with integer coefficients unless the coefficient module is mentioned.

A PL map $p : M \to B$ has Property $R \cong$ if, for each $b \in B$, a retraction $R : U \to p^{-1}b$ defined on some open set $U \supset p^{-1}b$ induces π_1-isomorphisms $(R|)_{\#} : \pi_1(p^{-1}b') \to \pi_1(p^{-1}b)$ for all $b' \in B$ sufficiently close to b.

2. Codimension 2 fibrators

In this section, we show that any cartesian product of a closed orientable n-manifold N_1 with finite fundamental group and a closed orientable asherical m-manifold N_2 with hopfian fundamental group, where $\chi(N_1)$ and $\chi(N_2)$ are nonzero, is a codimension 2 fibrator.

LEMMA 2.1 ([7], THEOREM 5.10). *All closed, hopfian manifolds with hopfian fundamental group and nonzero Euler characteristic are codimension 2 orientable fibrators.*

REMARK. Im and Kim [17] extended so that all closed, hopfian manifolds with hopfian fundamental group and nonzero Euler characteristic are codimension 2 fibrators.

LEMMA 2.2. *Let N_1^n be a closed orientable manifold with finite $\pi_1(N_1)$ and N_2^m be a closed orientable aspherical manifold. If $h : N_1 \times N_2 \to N_1 \times N_2$ is a degree one map, then so is $h_1 = pr \circ h \circ i : N_1 \to N_1$, where $pr : N_1 \times N_2 \to N_1$ is the projection and $i : N_1 \to N_1 \times N_2$ is the inclusion map.*

Proof. Assume that $h : N_1 \times N_2 \to N_1 \times N_2$ is a degree one map. By taking a universal covering space $(\tilde{N}_2, \theta)$ of N_2, $(N_1 \times \tilde{N}_2, id \times \theta)$ is a covering space of $N_1 \times N_2$. Let $\tilde{i} : N_1 \to N_1 \times \tilde{N}_2$ be a continuous map for which $(id \times \theta) \circ \tilde{i} = i$ and $\tilde{h} : N_1 \times \tilde{N}_2 \to N_1 \times \tilde{N}_2$ be a continuous map such that $h \circ (id \times \theta) = (id \times \theta) \circ \tilde{h}$ by the lifting property. The existence of $\tilde{h}$ follows from the fact that $\pi_1(N_2)$ is torsion-free and $(h \circ id \times \theta)_\#(\pi_1(N_1 \times \tilde{N}_2)) \subset (id \times \theta)_\#(\pi_1(N_1 \times \tilde{N}_2))$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
N_1 & \xrightarrow{\tilde{i}} & N_1 \times \tilde{N}_2 & \xrightarrow{\tilde{h}} & N_1 \times \tilde{N}_2 & \xrightarrow{q} & N_1 \\
{\scriptstyle id}\downarrow & & {\scriptstyle id\times\theta}\downarrow & & {\scriptstyle id\times\theta}\downarrow & & {\scriptstyle id}\downarrow \\
N_1 & \xrightarrow{i} & N_1 \times N_2 & \xrightarrow{h} & N_1 \times N_2 & \xrightarrow{pr} & N_1
\end{array}$$

where q is the projection from $N_1 \times \tilde{N}_2$ onto N_1. Because N_2 is aspherical, $\tilde{N}_2$ is contractible and then $H_i(\tilde{N}_2) = 0$ for all $i \geq 1$. According to the Künneth Theorem, we have $H_n(N_1 \times \tilde{N}_2) \cong H_n(N_1) \cong Z$ and

$H_n(N_1 \times N_2)$ is isomorphic to the direct sum of $\bigoplus_{i=0}^n H_{n-i}(N_1) \otimes H_i(N_2)$ and $\bigoplus_{i=0}^n H_{n-i-1}(N_1) * H_i(N_2)$. Thus, by the diagram chasing, it is easily checked that $h_*(H_n(N_1)) \subset H_n(N_1)$ when we restrict h_* to $H_n(N_1)$ and equate $H_n(N_1)$ with $H_n(N_1) \otimes H_0(N_2) \subset H_n(N_1 \times N_2)$. Rewrite $H_n(N_1 \times N_2)$ in a form $\{\text{free}\} \oplus \{\text{torsion}\}$. Note that $H_n(N_1 \times N_2)$ is finitely generated. Since h_* is an isomorphism, the restriction h_* to $\{\text{free}\}$ of $H_n(N_1 \times N_2)$ is an isomorphism and induces an invertible $(k \times k)$-matrix of the following form

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix}.$$

Here A_{11} is the matrix corresponding to map $h_*| : H_n(N_1) \to H_n(N_1)$ and A_{ij} is the matrix induced by the homomorphism from the i-th direct summand to the j-th direct summand of free part of $H_n(N_1 \times N_2)$. Because of $h_*(H_n(N_1)) \subset H_n(N_1)$, the restriction $h_*|\{\text{free}\}$ of h_* doesn't send the first factor $H_n(N_1)$ to any direct summand except itself and thus $A_{1j} = 0$ for $j = 2, \dots, k$. Since the isomorphism $h_*|\{\text{free}\}$ induces $detA = \pm 1$, we obtain $detA_{11} = \pm 1$. This implies $h_1 = pr \circ h \circ i$ is a degree one map. □

PROPOSITION 2.3. *Let N_1^n be a closed orientable manifold with finite $\pi_1(N_1)$ and N_2^m be a closed orientable aspherical manifold. Then $N_1 \times N_2$ is a hopfian manifold.*

Proof. Note that any closed orientable manifold with finite fundamental group is a hopfian manifold [13]. Let $h : N_1 \times N_2 \to N_1 \times N_2$ be a degree one map which induces a π_1-isomorphism.

To show that h is a homotopy equivalence, we consider homomorphisms

$$h_\# : \pi_i(N_1 \times N_2) \to \pi_i(N_1 \times N_2) \text{ for } i \geq 2.$$

Applying Lemma 2.2, the degree of $h_1 = pr \circ h \circ i : N_1 \to N_1$ is one. Since N_1 has finite fundamental group, h_1 induces a π_1-isomorphism. By the fact that N_1 is a hopfian manifold, h_1 is a homotopy equivalence and so is $\tilde{h}$ in the diagram of Lemma 2.2. Implying the fact that $(id \times \theta)_{\#} : \pi_i(N_1 \times \tilde{N}_2) \to \pi_i(N_1 \times N_2)$ is an isomorphism for $i \geq 2$, $h_{\#} : \pi_i(N_1 \times N_2) \to \pi_i(N_1 \times N_2)$ is an isomorphism for each $i \geq 2$. Therefore, h is a homotopy equivalence by the Whitehead theorem. □

Now, we state the main result in this section.

THEOREM 2.4. *Let N_1^n be a closed orientable manifold with finite $\pi_1(N_1)$ and N_2^m be a closed orientable aspherical manifold with hopfian $\pi_1(N_2)$, where where $\chi(N_1)$ and $\chi(N_2)$ are nonzero Euler characteristics. Then $N_1 \times N_2$ is a codimension 2 fibrator.*

Proof. From the fact [13] that any finitely generated group with finite index hopfian subgroup is hopfian, the fundamental group of $N_1 \times N_2$ is hopfian. Then, $N_1 \times N_2$ is a hopfian manifold with hopfian fundamental group by Proposition 2.3. Since the Euler characteristic of $N_1 \times N_2$ is nonzero, we have the conclusion from [17]. □

COROLLARY 2.5. *Let N_1^n be a closed orientable manifold with finite $\pi_1(N_1)$ and nonzero Euler characteristic, and N_2^m be a finite product of closed orientable surfaces with non-zero Euler characteristic. Then $N_1 \times N_2$ is a codimension 2 fibrator.*

Proof. Rewrite $N_1 \times N_2$ as $N_1' \times N_2'$, where N_1' is a product of N_1 and 2-spheres ,and N_2' is a product of closed orientable surfaces with negative Euler characteristic. Since N_2' is an aspherical manifold with nonzero Euler characteristic, $N_1 \times N_2$ is a codimension 2 fibrator from Theorem 2.4. □

REMARK. Let N_1^n be a closed, simply connected manifold with nonzero Euler characteristic, and N_2^m be a closed orientable aspherical manifold with hopfian fundamental group and nonzero Euler characteristic. Then $N_1 \times N_2$ is a codimension 2 fibrator.

3. Codimension k PL fibrators

Throughout this section, we restrict objects to the PL category and all manifolds are orientable. For a PL map $p : M \to B$, v will denote a vertex of a polyhedron B, $L = \text{link}(v, B), S = \text{star}(v, B) = v * L$, $L' = p^{-1}L$ and $S' = p^{-1}S$. These are understood to arise in the first barycentric subdivision of triangulations on which p is simplicial. Note that every codimension 2 fibrator N must be a codimension 2 PL fibrator.

LEMMA 3.1 ([8], Lemma 2.2). *If X is a CW-complex such that $\pi_i(X) = 0$ for $1 < i \leq k$ and if the map $f : X \to X$ induces an isomorphism $\pi_1(X) \to \pi_1(X)$, then f also induces isomorphisms*

$$f_* : H_i(X) \to H_i(X) \text{ and } f^* : H^i(X) \to H^i(X) \ (i \leq k)$$

LEMMA 3.2 ([9], Theorem 3.3'). *Suppose N^n is a closed hopfian manifold and $p : M^{n+k} \to B$ is an N-like PL map such that $H^n[p]$ is locally constant. Then p is an approximate fibration if and only if p has Property $R \cong$.*

To show the main result in this section, we begin with the following;

LEMMA 3.3. *Let G be a finite group and K be a cohopfian and torsion free group. Then $G \times K$ is cohopfian.*

Proof. Let $\phi : G \times K \to G \times K$ be a monomorphism. Consider the following diagram

$$\begin{array}{ccc} G & & G \\ {\scriptstyle i_G}\downarrow & & \uparrow{\scriptstyle pr_G} \\ G \times K & \xrightarrow{\phi} & G \times K \\ {\scriptstyle i_K}\uparrow & & \downarrow{\scriptstyle pr_K} \\ K & & K., \end{array}$$

where i_G, i_K are inclusions and pr_G, pr_K are projections.

From the fact that $pr_K \circ \phi \circ i_G(G)$ is trivial, $pr_K \circ \phi \circ i_K(K) = K$ and so $pr_K \circ \phi \circ i_K : K \to K$ is an isomorphism because K is cohopfian. Similarly, we have an isomorphism $pr_G \circ \phi \circ i_G : G \to G$, because $pr_K \circ \phi \circ i_G(G)$ is trivial. As a result, it is easy to see that ϕ is an isomorphism. $\square$

We state the main result in this section.

THEOREM 3.4. *Let N_1^n be a closed manifold with finite $\pi_1(N_1)$ and N_2^m be a closed aspherical manifold with hopfian fundamental group, where $\chi(N_1)$ and $\chi(N_2)$ are nonzero. If $\pi_i(N_1) = 0$ for $1 < i < k$, then $N_1 \times N_2$ is a codimension k PL fibrator.*

Proof. According to Theorm 2.4, $N_1 \times N_2$ is a codimension 2 PL fibrator. Let $p : M^{n+m+k} \to B^k$ be any N-like PL map, where $N \cong N_1 \times N_2$. We consider cases separately $k = 3$ and $k \geq 4$.

(Case 1.) $k = 3$

Because of $\chi(N) \neq 0$, B^3 is a 3-manifold [9, Lemma 2.3] and p has Property $R \cong$ [9, Lemma 5.1]. Since N is a codimension 2 PL fibrator, $p|L' : L' \to L$ is an approximate fibration. From the complete movability criterion [2], it suffices to show that $R : p^{-1}c \to p^{-1}v$ is a homotopy equivalence for any $c \in L$. Since N is a hopfian manifold with Property $R \cong$, it is enough to show that R is a degree one map.

Applying the fact that $\pi_2(N) = 0$ and p has Property $R \cong$, by Lemma 3.1 R induces isomorphisms

$$R_* : H_i(N) \to H_i(N) \text{ and } R^* : H^i(N) \to H^i(N) \ (i \leq 2).$$

First, the homology sequence of (S', L') shows

$$\cdots H_3(S', L') \to H_2(L') \to H_2(S') \to H_2(S', L') \cdots ,$$

where the first term is $H_3(S', L') \cong H^{n+m}(N) \cong Z$ and the last term is $H_2(S', L') \cong H^{n+m+1}(N) \cong 0$ by the Alexander duality.

Consider the following diagram

$$(*)\quad \begin{array}{ccccccc}
 & & H_2(p^{-1}c) & & & & \\
 & & \downarrow{\scriptstyle \lambda} & & & & \\
H_3(S', L') & \xrightarrow{\partial} & H_2(L') & \longrightarrow & H_2(S') & \longrightarrow & 0 \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p'_*} & & & & \\
H_3(S, L) & \xrightarrow{\cong} & H_2(L) & \longrightarrow & H_2(S) \cong 0 & &
\end{array}$$

Now, we show that $p'_* : H_2(L') \to H_2(L)(\cong Z)$ is an epimorphism. Since $p|L' : L' \to L$ is an approximate fibration, we have the homotopy exact sequence;

$$\cdots \to \pi_2(N) \to \pi_2(L') \to \pi_2(L) \to \pi_1(N) \to \cdots$$

By hypothesis, $\pi_2(N)$ is zero. Then $p_\# : \pi_2(L') \to \pi_2(L)$ is an isomorphism from the diagram and Property $R \cong$.

On the other hand, we have the natural following diagram

$$\begin{tikzcd}
\pi_2(L') \arrow[r, "\cong"] \arrow[d] & \pi_2(L) \arrow[d, "\cong"] \\
H_2(L') \arrow[r, "p'_*"] & H_2(L)
\end{tikzcd}$$

where the vertical isomorphism is because L is a 2-sphere. This shows that $p'_* : H_2(L') \to H_2(L)$ is an epimorphism.

Since S' collapses to $p^{-1}v$ and $R_* : H_2(p^{-1}c) \to H_2(p^{-1}v) \cong H_2(S')$ is an isomorphism, $H_2(L') \cong \mathrm{Im}\partial(H_2(N)) \oplus \mathrm{Im}\lambda(H_2(p^{-1}c))$ and we easily check that in (*), $p_* : H_3(S', L') \to H_3(S, L)$ is an isomorphism by the diagram chasing, and then we see that $p_* : H_3(S', S' - p^{-1}v) \to H_3(S, S - v)$ is an isomorphism. Similarly, we obtain an isomorphism $p_* : H_3(S', S' - p^{-1}c) \to H_3(S, S - c)$ for any $c \in S$ sufficiently close to v.

Then the following commutative diagram holds, where U is a connected open neighborhood of v in$S$ having compact closure and $c \in U$.

$$\begin{tikzcd}
H^{n+m}(p^{-1}v) \cong H_3(S', S' - p^{-1}v) \arrow[r, "\cong"] & H_3(S, S - v) \cong H^0(v) \\
H_3(S', S' - cl(p^{-1}U)) \arrow[r, "\cong"] \arrow[u] \arrow[d] & H_3(S, S - cl(U)) \cong H^0(clU) \arrow[u, "\cong"] \arrow[d, "\cong"] \\
H^{n+m}(p^{-1}c) \cong H_3(S', S' - p^{-1}c) \arrow[r, "\cong"] & H_3(S, S - c) \cong H^0(c)
\end{tikzcd}$$

This implies that R^* on the cohomology is constantly 1 near v, and implies the same on the homology by the universal coefficient theorem.

As a consequence, we conclude that $R : p^{-1}c \to p^{-1}v$ is a degree one map and N is a codimension 3 PL fibrator.

(Case 2.) $k \geq 4$

By induction, we can assume that N is a codimension $(k-1)$-fibrator. Let $p : M^{n+m+k} \to B^k$ be any PL N-like map. Then $p|L' : L' \to L$ is an approximate fibration and we have the homotopy exact sequence;

$$\cdots \to \pi_{k-1}(N) \to \pi_{k-1}(L') \to \pi_{k-1}(L) \to \pi_{k-2}(N) \to \cdots .$$

Note that $\pi_1(N_2)$ is cohopfian [12] and has no abelian normal subgroup [18]. Since $\pi_2(N) = 0$ and $\pi_2(L') \cong \pi_2(N) = 0$ by the general position [9], $\pi_2(L) \to \pi_1(N)$ in the homotopy exact sequence is a monomorphism. By the fact that $\pi_1(N_1)$ is finite and $\pi_1(N_2)$ is cohopfian with no abelian normal subgroup, it is easy to check from the homotopy exact sequence that $\pi_2(L)$ and $\pi_1(L)$ are finite. Since $\pi_i(N) = 0$ for $1 < i \leq k-1$ and $\pi_i(L') \cong \pi_i(S')$ for $i \leq k-2$ [9, Lemma 2.4], we have $\pi_i(L) = 0$ for $2 < i < k-1$. Consider the universal covering space $\tilde{L}$ of L. Then $\pi_1(\tilde{L}) = 0$, $\pi_2(\tilde{L})$ finite, and $\pi_i(\tilde{L}) = 0$ for $2 < i < k-1$. Hence $\pi_2(\tilde{L}) = 0$ by [8, Lemma 2.9], and so is $\pi_2(L)$. By Lemma 3.3 $\pi_1(N)$ is cohopfian, and so p has Property $R\cong$ and $\pi_1(L) = 0$. This implies that L^{k-1} is a homotopy $(k-1)$-sphere.

Repeating the method of case 1, $H^n[p]$ is locally constant. By Lemma 3.2, we have the conclusion that $p : M^{n+m+k} \to B^k$ is an approximate fibration. □

Corollary 3.5. *Let N^n be a closed aspherical manifold with hopfian fundamental group and nonzero Euler characteristic. Then N is a PL fibrator.*

Proof. Since N is a hopfian manifold with hyperhopfian group, N is a codimension 2 PL fibrator [7]. By copying the proof of Theorem 3.4, it is shown that N is a codimension k PL fibrator for all $k > 0$ so that N is a PL fibrator. □

Corollary 3.6. *Let N^n be a finite product of closed orientable surfaces with negative Euler characteristic. Then N is a PL fibrator.*

Proof. Since N is closed aspherical manifold with hopfian fundamental group and nonzero Euler characteristic, the conclusion follows from Corollary 3.6. □

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