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MARKOV-BERNSTEIN TYPE INEQUALITIES FOR POLYNOMIALS

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ABSTRACT. Let $\mu(x)$ be an increasing function on the real line with finite moments of all orders. We show that for any linear operator T on the space of polynomials and any integer $n \geq 0$, there is a constant $\gamma_n(T) \geq 0$, independent of $p(x)$, such that

$$\|Tp\| \leq \gamma_n(T)\,\|p\|,$$

for any polynomial $p(x)$ of degree $\leq n$, where

$$\|p\| = \left\{\int_{-\infty}^{\infty} |p(x)|^2\, d\mu(x)\right\}^{\frac{1}{2}}.$$

We find a formula for the best possible value $\Gamma_n(T)$ of $\gamma_n(T)$ and estimations for $\Gamma_n(T)$. We also give several illustrating examples when T is a differentiation or a difference operator and $d\mu(x)$ is an orthogonalizing measure for classical or discrete classical orthogonal polynomials.

1. Introduction

Markov-Bernstein type inequalities in weighted L^p spaces are interesting in themselves and important in approximation theory (see [1,4,7,9]). For example, consider an L^2-norm on the space $\mathcal{P}$ of polynomials with complex coefficients given by

$$\|p\|_{L^2(a,b;\omega)} := \left\{\int_a^b |p(x)|^2\omega(x)\, dx\right\}^{\frac{1}{2}},$$

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where $\omega(x)$ is an integrable function on (a, b), $-\infty \le a < b \le \infty$, such that $\omega(x) > 0$ on (a, b) and all moments

$$\omega_n := \int_a^b x^n \omega(x)\, dx, \quad n \ge 0$$

are finite. Then, using the orthonormal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$ with respect to the positive measure $\omega(x)dx$, Mirsky [9] showed that there exists a constant $\gamma_n = \gamma_n(a, b, \omega)$ such that

$$\|p'\|_{L^2(a,b,;\omega)} \le \gamma_n \|p\|_{L^2(a,b;\omega)}, \quad p \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the space of polynomials of degree $\le n$. Furthermore, the best constant Γ_n of γ_n satisfies

$$\Gamma_n := \sup_{p \in \mathcal{P}_n} \{\|p'(x)\|_{L^2(a,b;\omega)} \mid \|p\|_{L^2(a,b;\omega)} = 1\} \le \left\{ \sum_{k=1}^{n} k \|P_k'\|^2 \right\}^{\frac{1}{2}}.$$

In 1987, Dörfler [4] extended Mirsky's inequality to higher order derivatives and suggested a way to find the best constant involved. Guessab and Milovanovic [7] have found the best constants for higher order derivatives when $\omega(x)$ is a weight for classical orthogonal polynomials.

In this paper, we show that Markov-Bernstein type inequalities in weighted L^2-spaces hold not only for derivatives but also for any linear operator in $\mathcal{P}$ even when the measure $w(x)dx$ is replaced by any positive Borel measure $d\mu(x)$. We also give another way to find the best constant involved, which is easier to apply than Dörfler's. In particular, we obtain discrete versions of Markov-Bernstein type inequalities concerning for difference operators and compute the best constants involved in case the measure is the one for discrete classical orthogonal polynomials. Finally we give a similar result for linear operator on the space of polynomials with non-negative coefficients, which extends slightly the recent result by Chen [2].

2. Main results

We denote the degree of a polynomial $p(x)$ by $\deg(p)$ with the convention that $\deg(0) = -1$.

DEFINITION 2.1. A sequence of polynomials $\{P_n(x)\}_{n=0}^N$ is called an orthonormal polynomial system (ONPS) of order N ($N \geq 1$ is an integer or ∞) if $\deg(P_n) = n$, $0 \leq n \leq N$ and there exists an increasing function $\mu(x)$ on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)\,d\mu(x) = \delta_{mn}, \quad 0 \leq m, n \leq N,$$

where δ_{mn} is the Kronecker delta.

From now on, we always assume that $\mu(x)$ is an increasing function on $\mathbb{R}$ such that

$$\left|\int_{-\infty}^{\infty} x^n\,d\mu(x)\right| < \infty, \quad n = 0, 1, 2, \cdots.$$

We set

$$spec(\mu) := \{x \in \mathbb{R} \mid \mu(x+\epsilon) - \mu(x-\epsilon) > 0, \quad \text{for any } \epsilon > 0\}$$

and let $N+1$ be the cardinality of the set $spec(\mu)$. Then there exists a unique ONPS $\{P_n(x)\}_{n=0}^N$ of order N relative to $d\mu(x)$ (see [11]).

We use the following notations: for any a with $1 \leq a \leq \infty$,

$$\|c\|_a := \left(\sum_{k=0}^{n} |c_k|^a\right)^{\frac{1}{a}} \quad \text{if } 1 \leq a < \infty, \quad \|c\|_\infty := \max\{|c_k| : 0 \leq k \leq n\}$$

for any vector $c = (c_0, c_1, \cdots, c_n)$ in $\mathbb{C}^{n+1}$ and

$$\|p\| := \left\{\int_{-\infty}^{\infty} |p(x)|^2\,d\mu(x)\right\}^{\frac{1}{2}}, \quad p \in \mathcal{P}.$$

We note that on $\mathcal{P}_n$, $\|p\|$ is a norm if $0 \leq n \leq N$ and a semi-norm if $n > N$.

Let T be any linear operator from $\mathcal{P}_n$ into $\mathcal{P}$, where n is any fixed integer with $0 \leq n \leq N$. Then T is bounded so that there is a constant $\gamma_n(T)$, depending only on n and T, such that

$$\|Tp\| \leq \gamma_n(T)\|p\|, \quad p \in \mathcal{P}_n. \tag{2.1}$$

We let $\Gamma_n(T)$ be the smallest possible value of $\gamma_n(T)$ in (2.1). That is, $\Gamma_n(T)$ is the operator norm of T : $\Gamma_n(T) := \sup_{\|p\|=1} \|Tp\|$.

In the following, we let $\{\phi_k(x)\}_{k=0}^\infty$ be any sequence of polynomials with $\deg(\phi_k) = k$, $k \geq 0$, that is, $\{\phi_k(x)\}_{k=0}^\infty$ is a basis of $\mathcal{P}$.

THEOREM 2.1. *Let T and $\Gamma_n(T)$ be the same as above. Then*

$$\Gamma_n(T) = \sup_{c\in\mathbb{C}^{n+1}\setminus\{0\}} \sqrt{\frac{cD^T A(m)\bar{D}\bar{c}^T}{cA(n)\bar{c}^T}},$$

where $c = (c_0, c_1, \cdots, c_n)$ are vectors in $\mathbb{C}^{n+1}$ and $D = (d_k^j)_{j=0,k=0}^{m\ \ n}$ and $A(n) = (a_{ij})_{i,j=0}^n$ are matrices whose entries are given by

$$(T\phi_k)(x) := \sum_{j=0}^m d_k^j \phi_j(x), \quad m := \max_{0\le k\le n} \deg(T\phi_k)$$

and

$$a_{ij} := \int_{-\infty}^{\infty} \phi_i(x)\bar{\phi}_j(x)\, d\mu(x).$$

Proof. For any polynomial $p(x) \in \mathcal{P}_n$, we may write it as

$$p(x) = \sum_{k=0}^n c_k \phi_k(x).$$

Then, we have

$$\|p\|^2 = \int_{\mathbb{R}} |p(x)|^2\, d\mu(x) = \sum_{j=0}^n \sum_{k=0}^n c_j \bar{c}_k \int_{\mathbb{R}} \phi_j(x)\bar{\phi}_k(x)\, d\mu(x) = cA(n)\bar{c}^T.$$

$$\begin{aligned}\|Tp\|^2 &= \int_{\mathbb{R}} |Tp(x)|^2\, d\mu(x) = \int_{\mathbb{R}} \sum_{k=0}^n \left(c_k \sum_{j=0}^m d_k^j \right) \phi_j \sum_{\ell=0}^n \left(\bar{c}_\ell \sum_{i=0}^m \bar{d}_\ell^i \right) \bar{\phi}_i\, d\mu(x) \\ &= \sum_{j=0}^m \left(\sum_{k=0}^n c_k d_k^j \right) \sum_{i=0}^m \left(\sum_{\ell=0}^n \bar{c}_\ell \bar{d}_\ell^i \right) \int_{\mathbb{R}} \phi_j \bar{\phi}_i\, d\mu(x) \\ &= (Dc^T)^T A(m)(\bar{D}\bar{c}^T).\end{aligned}$$

Hence, we have

$$\Gamma_n(T) = \sup_{p\in\mathcal{P}_n\setminus\{0\}} \frac{\|Tp\|}{\|p\|} = \sup_{c\in\mathbb{C}^{n+1}\setminus\{0\}} \sqrt{\frac{cD^T A(m)\bar{D}\bar{c}^T}{cA(n)\bar{c}^T}}.$$

□

THEOREM 2.2. *Let T and $\Gamma_n(T)$ be the same as in* Theorem 2.1. *If we assume $m = \max_{0 \le k \le n} \deg(TP_k) \le N$, then*

$$\Gamma_n(T) = \sup_{\substack{\|c\|_2 = 1 \\ c \in \mathbb{C}^{n+1}}} \left\{ \sum_{j=0}^{m} \left| \sum_{k=0}^{n} c_k d_k^j \right|^2 \right\}^{\frac{1}{2}}. \tag{2.2}$$

Here, $D = (d_k^j)_{j=0, j=k}^{m \quad n}$ is the matrix whose entries are given by

$$(TP_k)(x) = \sum_{j=0}^{m} d_k^j P_j(x), \tag{2.3}$$

where $\{P_n(x)\}_{n=0}^N$ is the ONPS relative to $d\mu(x)$. Moreover, $\Gamma_n(T)$ satisfies an estimate : for any a with $1 \le a \le \infty$

$$\max_{0 \le k \le n} \|TP_k\| \le \Gamma_n(T) \le C(a, n)\ \|(\|TP_0\|, \cdots, (\|TP_n\|)\|_b, \tag{2.4}$$

where

$$C(a, n) = \sup_{c \in \mathbb{C}^{n+1} \setminus \{0\}} \frac{\|c\|_a}{\|c\|_2} \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} = 1.$$

Proof. If we take $\{P_k(x)\}_{k=0}^N$ for $\{\phi_k(x)\}_{k=0}^N$ in the proof of Theorem 2.1, then the matrix $A(n)$, $n \ge 1$, becomes an identity matrix since $\{P_k(x)\}_{k=0}^N$ is an ONPS relative to $d\mu(x)$. Hence, $cA(n)\bar{c}^T = \|c\|^2$ and so

$$\Gamma_n(T) = \sup_{\substack{\|c\|_2 = 1 \\ c \in \mathbb{C}^{n+1}}} \sqrt{cD^T \bar{D} \bar{c}^T},$$

which is the matrix form of the equation (2.2). On the other hand, for any a with $1 \le a \le \infty$ and for any $p(x) = \sum_{k=0}^{n} c_k P_k(x)$ in $\mathcal{P}_n$, Hölder's inequality implies

$$\|Tp\| \le \sum_{k=0}^{n} |c_k|\, \|TP_k\| \le \|c\|_a\, \|(\|TP_0\|, \cdots, \|TP_n\|)\|_b \quad \left(\frac{1}{a} + \frac{1}{b} = 1\right).$$

Now, let $S : \mathcal{P}_n \longrightarrow \mathbb{C}^{n+1}$ be the linear operator defined by

$$S(p) = S\left(\sum_{k=0}^{n} c_k P_k\right) = (c_0, c_1, \cdots, c_n).$$

Then

$$\|c\|_a = \|S(p)\|_a \leq \|S\| \, \|p\| = c(a,n)\|p\|.$$

Hence,

$$\|Tp\| \leq C(a,n) \, \|(\|TP_0\|, \cdots, \|TP_n\|)\|_b \, \|p\|, \qquad p \in \mathcal{P}_n,$$

which gives the upper bound for $\Gamma_n(T)$ in (2.4). The lower bound for $\Gamma_n(T)$ in (2.4) is trivial since $\|P_k\| = 1$, $0 \leq k \leq n$. □

Of particular interest to us are the cases when $a = 1$ or 2.

COROLLARY 2.3. *Let T and $\Gamma_n(T)$ be the same as in* Theorem 2.2. *Then*

$$\max_{0\leq k\leq n} \|TP_k\| \leq \Gamma_n(T) \leq \sqrt{n+1} \max_{0\leq k\leq n} \|TP_k\|. \tag{2.5}$$

Proof. If we set $a = 1$ in (2.4), then we obtain

$$\max_{0\leq k\leq n} \|TP_k\| \leq \Gamma_n(T) \leq C(1,n) \max_{0\leq k\leq n} \|TP_k\|. \tag{2.6}$$

Since $\|c\|_1 \leq \sqrt{n+1}\|c\|_2$, $c \in \mathbb{C}^{n+1}$,

$$C(1,n) \leq \sqrt{n+1}$$

so that (2.5) follows from (2.6). □

When $a = 2$, we obtain:

COROLLARY 2.4. *Let T and $\Gamma_n(T)$ be the same as in* Theorem 2.2. *Then*

$$\max_{0\le k\le n}\|TP_k\| \le \Gamma_n(T) \le \left\{\sum_{k=0}^{n}\|TP_k\|^2\right\}^{\frac{1}{2}}. \tag{2.7}$$

Dörfler [4] obtained the best constant $\Gamma_n(T)$ (in a different form) and the inequality (2.7) when $\mu(x)$ is absolutely continuous so that $d\mu(x) = w(x)dx$, $N = \infty$, and $T = \frac{d^r}{dx^r}$ (see also Mirsky [9] in case $r = 1$).

Since $\Gamma_n(T)$ is the smallest value of λ satisfying

$$\frac{cD^TA(m)\bar{D}\bar{c}^T}{cA(n)\bar{c}^T} \le \lambda^2, \quad c \in \mathbb{C}^{n+1}\setminus\{0\}, \tag{2.8}$$

$\Gamma_n(T)$ is the smallest constant λ such that $\lambda^2A(n) - D^TA(m)\bar{D}$ is positive-semi definite. By the positive-definiteness of $d\mu(x)$ on $\mathcal{P}_n$, $0 \le n \le N$, $A(n)$ is Hermitian for $n \ge 0$ and positive-definite for $0 \le n \le N$ and $D^TA(m)\bar{D}$ is Hermitian and positive-semidefinite for $m \ge 0$.

If $A(n)$ and $D^TA(m)\bar{D}$ commute, then they have $n+1$ common linearly independent eigenvectors $\{u_i\}_{i=0}^n$ such that

$$Au_i = \mu_i u_i, \quad D^TA\bar{D}u_i = \nu_i u_i, \quad i = 0, 1, \cdots, n \tag{2.9}$$

since both $A(n)$ and $D^TA(m)\bar{D}$ are Hermitian (see [8]).

Now, we have the following.

THEOREM 2.5. *Let T and $\Gamma_n(T)$ be the same as in* Theorem 2.1. *If $A(n)$ and $D^TA(m)\bar{D}$ commute, then*

$$\Gamma_n(T) = \max_{0\le i\le n}\sqrt{\frac{\nu_i}{\mu_i}},$$

where μ_i (> 0) and ν_i (≥ 0), $0 \le i \le n$, are the eigenvalues of $A(n)$ and $D^TA(m)\bar{D}$ respectively as in (2.9).

Proof. Let $\sigma := \max_{0\le i\le n} \sqrt{\frac{\nu_i}{\mu_i}}$. Then, $\sigma^2 A(n) - D^T A(m)\bar{D}$ is positive-semidefinite since it is Hermitian and has non-negative eigenvalues $\sigma^2\mu_i - \nu_i$, $0 \le i \le n$. Hence, $\Gamma_n(T) \le \sigma$. Conversely, the left hand side of the inequality (2.8) becomes σ^2 when $c = \bar{u}_r$, where r is an integer such that $0 \le r \le n$ and $\sigma^2 = \frac{\nu_r}{\mu_r}$. Hence, $\Gamma_n(T) \ge \sigma$. □

In particular, if $m \le N$ and if we take $\{\phi_k(x)\}_{k=0}^N = \{P_k(x)\}_{k=0}^N$ so that $A(n)$ is the identity matrix, then $\Gamma_n(T)^2$ is equal to the largest eigenvalue of the matrix $D^T\bar{D}$, i.e., the largest singular value of D (D as in Theorem 2.2). This fact is observed and used by Dörfler [4] when T is a differentiation and $N = \infty$.

Below we give several examples illustrating Theorem 2.2 when T is a differential or difference operator and $d\mu(x)$ is an orthogonalizing measure for classical or discrete classical orthogonal polynomials.

Example 2.1. Consider $d\mu(x) = x^\alpha e^{-x} H(x)\,dx$, where $\alpha > -1$ and $H(x)$ is the Heaviside step function. The corresponding ONPS $\{P_n(x)\}_{n=0}^\infty$ is

$$P_n(x) = \left\{\frac{n!}{\Gamma(n+\alpha+1)}\right\}^{\frac{1}{2}} L_n^{(\alpha)}(x), \quad n \ge 0,$$

where $L_n^{(\alpha)}(x)$ is the n-th Laguerre polynomial (see [3, 11]). From the addition formula (see [11, p. 391, Problem 90])

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_{n-k}^{(\alpha)}(x) L_k^{(\beta)}(y),$$

we can easily deduce for $0 \le k \le n$

$$\frac{d^r}{dx^r} L_n^{(\alpha)}(x) = (-1)^r L_{n-r}^{(\alpha+r)}(x) = (-1)^r \sum_{j=0}^{n-r} \binom{j+r-1}{j} L_{n-r-j}^{(\alpha)}(x)$$

and
(2.10)

$$\frac{d^r}{dx^r}P_n(x)$$
$$= (-1)^r\left\{\frac{n!}{\Gamma(n+\alpha+1)}\right\}^{\frac{1}{2}}\sum_{j=0}^{n-r}\left(\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}\right)^{\frac{1}{2}}\binom{n-j-1}{r-1}P_j(x).$$

If we take $T = \frac{d^r}{dx^r}$, then the relation (2.10) implies that the coefficients in (2.3) are given by

$$d_k^j = \begin{cases} (-1)^r\left\{\frac{k!}{j!}\frac{\Gamma(j+\alpha+1)}{\Gamma(k+\alpha+1)}\right\}^{\frac{1}{2}}\binom{k-j-1}{k-j-r}, & 0 \le j \le k-r \\ 0, & \text{otherwise} \end{cases}$$

so that

$$\Gamma_n\left(\frac{d^r}{dx^r}\right) = \sup_{\substack{c\in\mathbb{C}^{n+1}\\ \|c\|_2=1}}\left\{\sum_{j=0}^{n-r}\left|\sum_{k=j+r}^{n} c_k d_k^j\right|^2\right\}^{\frac{1}{2}}.$$

In general, it's very hard to compute $\Gamma_n(\frac{d^r}{dx^r})$ explicitly. However, we have

$$\max_{r\le k\le n}\|\frac{d^r}{dx^r}P_k\|$$
$$= \max_{r\le k\le n}\left\|(-1)^r\left(\frac{k!}{\Gamma(k+\alpha+1)}\right)^{\frac{1}{2}}\times\right.$$
$$\left.\sum_{j=0}^{k-r}\left(\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}\right)^{\frac{1}{2}}\binom{k-j-1}{r-1}P_j(x)\right\|$$
$$= \max_{r\le k\le n}\left\{\frac{k!}{\Gamma(k+\alpha+1)}\right\}^{\frac{1}{2}}\left\{\sum_{j=0}^{k-r}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}\binom{k-j-1}{r-1}^2\right\}^{\frac{1}{2}}.$$

Since $\frac{k!}{\Gamma(k+\alpha+1)}$ is decreasing (increasing) if $\alpha \geq 0$ $(-1 < \alpha < 0)$,

$$\begin{cases} \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n \leq \max\limits_{r\leq k\leq n} \|\frac{d^r}{dx^r}P_k\| \leq \left(\frac{r!}{\Gamma(r+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n, \\ \text{if } \alpha \geq 0 \\ \left(\frac{r!}{\Gamma(r+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n \leq \max\limits_{r\leq k\leq n} \|\frac{d^r}{dx^r}P_k\| \leq \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n, \\ \text{if } -1 < \alpha < 0, \end{cases}$$

where

$$\Delta_n = \left\{\sum_{j=0}^{n-r} \frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)} \binom{n-j-1}{r-1}^2\right\}^{\frac{1}{2}}.$$

Hence the inequality (2.5) gives

$$\text{(2.11)} \quad \begin{cases} \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n \leq \Gamma_n\left(\frac{d^r}{dx^r}\right) \leq \left(\frac{r!(n+1)}{\Gamma(r+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n, \\ \text{if } \alpha \geq 0 \\ \left(\frac{r!}{\Gamma(r+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n \leq \Gamma_n\left(\frac{d^r}{dx^r}\right) \leq \left(\frac{n!(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} \Delta_n, \\ \text{if } -1 < \alpha < 0. \end{cases}$$

In particular, when $\alpha = 0$, we obtain
(2.12)

$$\left\{\sum_{j=0}^{n-r} \binom{n-j-1}{r-1}^2\right\}^{\frac{1}{2}} \leq \Gamma_n\left(\frac{d^r}{dx^r}\right) \leq \sqrt{n+1}\left\{\sum_{j=0}^{n-r} \binom{n-j-1}{r-1}^2\right\}^{\frac{1}{2}}.$$

When $r = 1$, Dörfler [6] obtained a sharper upper bound for $\Gamma_n(d/dx)$ than (2.11). When $\alpha = 0$, Dörfler [5, Theorem 2] obtained a similar estimation as (2.12) which gives a sharper upper bound. But, the method in [5] is not easy to extend for $\alpha > -1$ but $\alpha \neq 0$. In particular, when $r = 1$ and $\alpha = 0$, Turán [12] found explicitly $\Gamma_n(d/dx)$ as

$$\Gamma_n\left(\frac{d}{dx}\right) = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}.$$

Now, consider the forward difference operator Δ defined by

$$\Delta p(x) = p(x+1) - p(x), \quad p \in \mathcal{P}.$$

EXAMPLE 2.2. Let $\nu(x)$ be a step function such that $spec(\nu) = \{0, 1, 2, \cdots\}$ and jump of ν at $x = n$ is $\frac{\mu^n e^{-\mu}}{n!}$, $n \geq 0$, where $\mu > 0$. Then, the corresponding ONPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to $d\nu(x)$ is

$$P_n(x) = \sqrt{\frac{\mu^n}{n!}}\, C_n^{(\mu)}(x), \quad n \geq 0,$$

where $C_n^{(\mu)}(x)$ is the n-th Charlier polynomial (see [10]). From the relation $\Delta C_n^{(\mu)}(x) = -\frac{n}{\mu} C_{n-1}^{(\mu)}(x)$, it can be easily shown that

$$\Delta^r P_k(x) = (-1)^r \left\{ \mu^{-r} r! \binom{k}{r} \right\}^{\frac{1}{2}} P_{k-r}(x), \quad k \geq r. \tag{2.13}$$

If we take $T = \Delta^r$ $(1 \leq r \leq n)$, then the relation (2.13) implies that the coefficients d_k^j in (2.3) are given by

$$d_k^j = \begin{cases} (-1)^r (\mu^{-r} r! \binom{k}{r})^{\frac{1}{2}}, & j = k - r \\ 0, & \text{otherwise} \end{cases}$$

so that

$$\Gamma_n(\Delta^r) = \sup_{\substack{c \in \mathbb{C}^{n+1} \\ \|c\|_2 = 1}} \left(\sum_{j=0}^{n-r} \Big| \sum_{k=0}^{n} c_k d_k^j \Big|^2 \right)^{\frac{1}{2}} = \sup_{\substack{c \in \mathbb{C}^{n+1} \\ \|c\|_2 = 1}} \left(\mu^{-r} r! \sum_{k=r}^{n} \binom{k}{r} |c_k|^2 \right)^{\frac{1}{2}}$$

$$= \left\{ \mu^{-r} r! \binom{n}{r} \right\}^{\frac{1}{2}}.$$

Moreover, it is easy to see that the supremum is attained only when $c = (0, 0, \cdots, 0, b)$, $|b| = 1$ by using the Lagrange multiplier. Thus, we have

$$\|\Delta^r p\| \leq \left\{ \mu^{-r} r! \binom{n}{r} \right\}^{\frac{1}{2}} \|p\|, \quad p \in \mathcal{P}_n$$

and equality holds if and only if $p(x) = aP_n(x)$, where a is a constant.

Dörfler [4] obtained a similar result as Example 2.2 for the operator $\frac{d^r}{dx^r}$ and the measure $e^{-x^2}dx$ using Hermite polynomials, which are continuous analogues of Charlier polynomials.

EXAMPLE 2.3. Let $\nu(x)$ be a step function such that $spec(\nu) = \{0, 1, 2, \cdots\}$ and jump of $\nu(x)$ at $x = n$ is $\frac{\mu^n \Gamma(\gamma+n)}{n!\Gamma(\gamma)}$, $n \geq 0$, where $\gamma > 0$, $0 < \mu < 1$. Then, the corresponding ONPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to $d\nu(x)$ is

$$P_n^{(\gamma,\mu)}(x) = \left\{ \frac{\mu^n (1-\mu)^\gamma}{n!(\gamma)_n} \right\}^{\frac{1}{2}} M_n^{(\gamma,\mu)}(x), \quad n \geq 0,$$

where $M_n^{(\gamma,\mu)}(x)$ is the n-th Meixner polynomial (see [10]). From the relation $\Delta M_n^{(\gamma,\mu)}(x) = -\frac{n(1-\mu)}{\mu} M_{n-1}^{(\gamma+1,\mu)}(x)$, it can be easily shown that

$$\Delta^r P_k^{(\gamma,\mu)}(x) = (-1)^r E(r, k, \gamma, \mu) P_{k-r}^{(\gamma+r,\mu)}(x),$$

where

$$E(r, k, \gamma, \mu) = \left\{ \frac{k!\Gamma(\gamma)(1-\mu)^r}{(k-r)!\Gamma(\gamma+r)\mu^r} \right\}^{\frac{1}{2}}.$$

Hence, we have

$$\begin{aligned}
&\|\Delta^r P_k^{(\gamma,\mu)}\|^2 \\
&= E^2(r, k, \gamma, \mu) \sum_{i=0}^{\infty} [P_{k-r}^{(\gamma+r,\mu)}(i)]^2 \frac{\mu^i \Gamma(\gamma+i)}{i!\Gamma(\gamma)} \\
&= E^2(r, k, \gamma, \mu) \sum_{i=0}^{\infty} [P_{k-r}^{(\gamma+r,\mu)}(i)]^2 \frac{\mu^i \Gamma(\gamma+r+i)}{i!\Gamma(\gamma+r)} \cdot \frac{\Gamma(\gamma+i)\Gamma(\gamma+r)}{\Gamma(\gamma+r+i)\Gamma(\gamma)} \\
&\leq E^2(r, k, \gamma, \mu) \max_{0 \leq i \leq \infty} \frac{\Gamma(\gamma+i)\Gamma(\gamma+r)}{\Gamma(\gamma+r+i)\Gamma(\gamma)} \\
&= E^2(r, k, \gamma, \mu),
\end{aligned}$$

since

$$\sum_{i=0}^{\infty} [P_{k-r}^{(\gamma+r,\mu)}(i)]^2 \frac{\mu^i \, \Gamma(r+\gamma+i)}{i! \, \Gamma(\gamma+r)} = 1, \quad k = r, r+1, \cdots,$$

and $\frac{\Gamma(\gamma+i)}{\Gamma(\gamma+r+i)}$, $i \geq 0$, is monotone decreasing in i. Therefore,

$$\|\Delta^r P_k^{(\gamma,\mu)}\| \leq \left\{ \frac{k!\Gamma(\gamma)(1-\mu)^r}{(k-r)!\Gamma(\gamma+r)\mu^r} \right\}^{\frac{1}{2}}.$$

The estimation (2.5) in Corollary 2.3 implies that

$$\begin{aligned}\Gamma_n(\Delta^r) &\leq \sqrt{n+1}\left\{ \frac{\Gamma(\gamma)(1-\mu)^r}{\Gamma(\gamma+r)\mu^r} \right\}^{\frac{1}{2}} \max_{r\leq k\leq n} \left\{ \frac{k!}{(k-r)!} \right\}^{\frac{1}{2}} \\ &= \sqrt{n+1}\left\{ \frac{\Gamma(\gamma)(1-\mu)^r}{\Gamma(\gamma+r)\mu^r} \right\}^{\frac{1}{2}} \left\{ \frac{n!}{(n-r)!} \right\}^{\frac{1}{2}}.\end{aligned}$$

Finally, we give a minor extension of the recent work by Chen [2], which handles the similar extremal problem on the space of polynomials with non-negative coefficients.

Assume that T is a linear operator on the space of real polynomials and consider the following extremal problem

$$\tilde{\Gamma}_n(T) := \sup_{p\in S_n} \frac{\|Tp\|}{\|p\|},$$

where

$$S_n := \{p(x) \in \mathcal{P}_n : p(x) = \sum_{k=0}^{n} c_k\phi_k(x), \quad c_k \geq 0, \quad 0 \leq k \leq n\}$$

and $\{\phi_k(x)\}_{k=0}^{\infty}$ is a sequence of real polynomials with $\deg(\phi_k) = k$, $k \geq 0$.

By the same arguments as before, we can see that $\tilde{\Gamma}_n(T)$ is the smallest value of λ such that

$$c[\lambda^2 A(n) - D^T A(m) D]c^T \geq 0 \tag{2.14}$$

for all $c = (c_0, c_1, \cdots, c_n)$ in $\mathbb{R}^{n+1}$ with $c_i \geq 0$, $0 \leq i \leq n$, where matrices $A(n)$ and D are the same as in Theorem 2.1.

We set $D^T A(m)D = (\tilde{a}_{ij})_{i,j=0}^n$ and assume that $a_{ij} \geq 0$, $\tilde{a}_{ij} \geq 0$ for $0 \leq i, j \leq n$ and if $a_{ij} = 0$, then $\tilde{a}_{ij} = 0$. Let

$$\tilde{\sigma} := \max_{0 \leq i,j \leq n} \left\{ \sqrt{\frac{\tilde{a}_{ij}}{a_{ij}}} \,\middle|\, a_{ij} > 0 \right\}.$$

Since c has all positive elements, the inequality (2.14) holds for $\lambda = \tilde{\sigma}$. Hence,

$$\tilde{\Gamma}_n(T) \leq \tilde{\sigma}. \tag{2.15}$$

THEOREM 2.6. *If $\tilde{\sigma}$ occurs at $i = j = r$, then $\tilde{\Gamma}_n(T) = \tilde{\sigma}$, that is,*

$$\|Tp\| \leq \sqrt{\frac{\tilde{a}_{rr}}{a_{rr}}}\, \|p\|, \quad p \in S_n \tag{2.16}$$

and equality holds for $p(x) = b\phi_r(x)$, where b is a non-negative constant.

Proof. For $c = (c_0, c_1, \cdots, c_n)$ with $c_i = 0$, $i \neq r$ and $c_r = 1$, we have

$$c(\tilde{\sigma}^2 A(n) - D^T A(m)D)c^T = 0 \tag{2.17}$$

so that $\tilde{\Gamma}_n(T) \geq \tilde{\sigma}$ and so $\tilde{\Gamma}_n(T) = \tilde{\sigma}$ by (2.15). Equality in (2.16) holds for any $p(x) = \sum_{k=0}^{n} c_k \phi_k(x)$ in S_n if and only if $c = (c_0, c_1, \cdots, c_n)$ satisfies (2.17), which holds, in particular, if $c_i = 0$ for $i \neq r$ and c_r (≥ 0) is arbitrary. □

EXAMPLE 2.4. Consider $d\mu(x) = (1 - x^2)^\alpha\, dx$ with $\alpha > -1$ and $\{\phi_k(x)\}_{k=0}^\infty = \{x^k\}_{k=0}^\infty$. Then we have

$$S_n = \{p(x) \in \mathcal{P}_n : p(x) = \sum_{k=0}^{n} c_k x^k, \quad c_k \geq 0, \quad 0 \leq k \leq n\}.$$

For $T = \frac{d^r}{dx^r}$ $(1 \le r \le n)$, we have

$$a_{ij} = \int_{-1}^{1} x^{i+j}(1-x^2)^\alpha \, dx = \frac{1-(-1)^{i+j+1}}{2} B\Big(\frac{i+j+1}{2}, \alpha+1\Big),$$

$$\tilde{a}_{ij} = \begin{cases} 0, & \text{if } i<r \text{ or } j<r \\ \int_{-1}^{1} \frac{i!}{(i-r)!}\frac{j!}{(j-r)!} x^{i+j-2r}(1-x^2)^\alpha \, dx = \frac{i!}{(i-r)!}\frac{j!}{(j-r)!} a_{i-r,j-r} & \\ & \text{if } i,j \ge r, \end{cases}$$

where $B(\cdot,\cdot)$ is the Beta function. Hence, we have $a_{ij} \ge 0$, $\tilde{a}_{ij} \ge 0$ and $\tilde{a}_{ij} = 0$ if $a_{ij} = 0$ and

$$\begin{aligned} \frac{\tilde{a}_{ij}}{a_{ij}} &= \frac{i!}{(i-r)!}\frac{j!}{(j-r)!}\frac{B(\frac{i+j-2r+1}{2}, \alpha+1)}{B(\frac{i+j+1}{2}, \alpha+1)} \\ &= \prod_{k=0}^{r-1} \frac{(i-k)(j-k)(2\alpha+i+j-2k+1)}{i+j-2k-1} \qquad (a_{ij} \ne 0). \end{aligned}$$

Since each factor $\frac{(i-k)(j-k)(2\alpha+i+j-2k+1)}{i+j-2k-1}$, $0 \le k \le r-1$, increases with i and j, $\frac{\tilde{a}_{ij}}{a_{ij}}$ attains its maximum when $i = j = n$. Hence,

$$\tilde{\sigma}^2 = \frac{\tilde{a}_{nn}}{a_{nn}} = \left[\frac{n!}{(n-r)!}\right]^2 \prod_{k=0}^{r-1} \frac{2\alpha+2n-2k+1}{2n-2k-1}.$$

By Theorem 2.6, we have

$$\|p^{(r)}\| \le \frac{n!}{(n-r)!}\left\{\prod_{k=0}^{r-1} \frac{2\alpha+2n-2k+1}{2n-2k-1}\right\}^{\frac{1}{2}} \|p\|, \quad p \in S_n$$

and equality holds when $p(x) = bx^n$.

Similar results for $T = \frac{d^r}{dx^r}$ can be obtained for $d\mu(x) = x^\alpha e^{-x}\,dx$ $(\alpha > -1)$ and $e^{-\alpha x^2}$ $(\alpha > 0)$ as Chen [2] did only for $T = d/dx$.

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