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ON POLY-EULERIAN NUMBERS

JIN-WOO SON AND MIN-SOO KIM

ABSTRACT. In this paper we define poly-Euler numbers which generalize ordinary Euler numbers. We construct a p-adic poly-Euler measure by the poly-Euler polynomials and derive an integral formula.

1. Introduction

In the case of Euler number, we consider the coefficients of the expansion of $\frac{2}{e^t+1}$ and $\frac{1}{\cosh t}$:

$$e^{Ht} = \frac{2}{e^t+1} = \sum_{k=0}^{\infty} H_k \frac{t^k}{k!} \text{ and } e^{Et} = \frac{1}{\cosh t} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}$$

where the symbols H_k and E_k are interpreted to mean that H^k(resp. E^k) must be replaced by H_k(resp. E_k) when we expand the one on the left. Here we see that $E_k = (2H+1)^k$.

The recurrence formula for the Euler numbers has the form $(E+1)^n + (E-1)^n = 0$, $E_0 = 1$. Thus, $E_{2n+1} = 0$, the E_{4n} are positive and E_{4n+2} are negative integers for all $n = 0, 1, \cdots$;

$$E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ E_{10} = -50521.$$

The Euler numbers are connected with the Bernoulli numbers. The Euler numbers are used in the summation of series.

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The generalized k-th Euler number $H_k(u)$ was studied by Frobenius (1910): for real algebraic u,

$$\frac{1-u}{e^t-u} = e^{H(u)t} = \sum_{m=0}^{\infty} \frac{H_m(u)}{m!} t^m. \tag{1}$$

Thus we have the relation

$$H_0(u) = 1, \ \ (H(u)+1)^k - uH_k(u) = 0 \quad (k \geq 1).$$

Consequently,

$$uH_k(u) = \sum_{j=0}^{k} \binom{k}{j} H_j(u) \ \text{ and } \ H_k(u) = \frac{1}{u-1} \sum_{j=0}^{k-1} \binom{k}{j} H_j(u),$$
$$\text{for } u \neq 1.$$

In this paper, we define poly-Euler numbers which generalize ordinary Euler numbers by Frobenius. We construct a p-adic poly-Euler measure by the poly-Euler polynomials and derive an important integral formula. In section 2, we define a sequence of the numbers $H_n^{(k)}$ $(n = 0, 1, 2, \cdots)$, which we say poly-Euler numbers and we give an explicit formula for $H_n^{(k)}$ in the real case. We prove the distribution relation for the poly-Euler polynomials. In section 3, the poly-Euler polynomials constructed in section 2 can be seen to be a p-adic poly-Euler measure. We obtain the poly-Euler numbers by using the integral formulas. In section 4, we study some properties of ordinary Euler numbers by Frobenius.

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2. Poly-Eulerian numbers

First we introduce the k-th *polylogarithm* defined by

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

for $k \geq 1$, $|z| < 1$ [1].

For parameter u with $u \in \mathbb{R}$ and $u > 1$, we have

$$Li_k(1 - e^{(1-u)}) = \sum_{m=1}^{\infty} \frac{(1 - e^{(1-u)})^m}{m^k} \quad \text{for } k \geq 1.$$

Thus

$$\begin{aligned}
&\frac{d}{du} Li_k(1 - e^{(1-u)}) \\
&= \sum_{m=1}^{\infty} \frac{m(1 - e^{(1-u)})^{m-1}}{m^k} e^{(1-u)} \\
&= \frac{e^{(1-u)}}{1 - e^{(1-u)}} \sum_{m=1}^{\infty} \frac{(1 - e^{(1-u)})^m}{m^{k-1}} \\
&= \frac{1}{e^{-(1-u)} - 1} Li_{k-1}(1 - e^{(1-u)}).
\end{aligned}$$

Hence $Li_k(1 - e^{(1-u)})$, $k \geq 2$ can be written in the form of iterated integrals:

$$\begin{aligned}
&Li_k(1 - e^{(1-u)}) \\
&= \int_1^u \frac{1}{e^{-(1-t)} - 1} Li_{k-1}(1 - e^{(1-t)}) dt \\
&= \int_1^u \frac{1}{e^{-(1-t)} - 1} \int_1^t \frac{1}{e^{-(1-t)} - 1} Li_{k-2}(1 - e^{(1-t)}) dt\, dt \\
&= \cdots \\
&= \underbrace{\int_1^u \frac{1}{e^{-(1-t)} - 1} \int_1^t \frac{1}{e^{-(1-t)} - 1} \cdots \int_1^t \frac{t-1}{e^{-(1-t)} - 1}}_{(k-1)-\text{times}} dt\, dt \cdots dt.
\end{aligned}$$

Now, we define the *poly-Euler number* $H_n^{(k)}(u)$ as

$$(2) \qquad \frac{Li_k(1 - e^{(1-u)})}{u - e^x} = \sum_{n=0}^{\infty} H_n^{(k)}(u) \frac{x^n}{n!} = e^{H^{(k)}(u)x} \quad \text{for } k \geq 1.$$

By comparing the coefficients of Taylor expansion on both side, we know that

$$(3)\qquad \begin{aligned} (H^{(k)}(u)+1)^n &= uH_n^{(k)}(u) \quad \text{for } \forall n \geq 1, \\ H_0^{(k)}(u) &= \frac{Li_k(1-e^{(1-u)})}{u-1}. \end{aligned}$$

The left hand side of (2) can be written as

$$\frac{1}{u-e^x}\underbrace{\int_1^u \frac{1}{e^{-(1-t)}-1}\int_1^t \frac{1}{e^{-(1-t)}-1}\cdots\int_1^t \frac{t-1}{e^{-(1-t)}-1}}_{(k-1)-\text{times}} dtdt\cdots dt$$

$$(4)\qquad = \sum_{n=0}^{\infty} H_n^{(k)}(u)\frac{x^n}{n!}.$$

where u is a real number with $u > 1$.

Also we define *poly-Euler polynomials* as

$$\frac{e^{xt}}{u-e^t}Li_k(1-e^{(1-u)}) = \sum_{n=0}^{\infty} H_n^{(k)}(u;x)\frac{t^n}{n!}.$$

Then we have

$$(5)\qquad (H^{(k)}(u)+x)^n = H_n^{(k)}(u;x), \ \ \forall n \geq 0,\ k \geq 1.$$

For $n \geq 0,\ k \geq 1$, we can write poly-Euler polynomials as

$$H_n^{(k)}(u;x) = \sum_{l=0}^{n} \binom{n}{l} H_l^{(k)}(u)x^{n-l}.$$

If $k = 1$, then $H_n^{(1)}(u) := H_n(u)$ is the ordinary Euler number by Frobenius.

The Stirling numbers of the second kind $S(n,m)$ $(n \geq 0,\ 0 \leq m \leq n)$ is defined by the formula

$$(6)\qquad x^n = \sum_{m=0}^{n} S(n,m)(x)_m,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(x)_0 = 1$. By simple calculation, the Stirling numbers of the second kind satisfies the following formulas (when $n=0$, the identity $0^0 = 1$ is understood):

$$S(n,m) = \frac{(-1)^m}{m!}\sum_{l=0}^{m}\binom{m}{l}(-1)^l l^n$$

and

$$\frac{(e^{(1-u)}-1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m)\frac{(1-u)^n}{n!}, \tag{7}$$

where $1 < u < 2$.

By (7)

$$\begin{aligned}
&\frac{Li_k(1-e^{(1-u)})}{u-e^x} \\
&= \frac{1}{u-e^x}\sum_{m=0}^{\infty}\frac{(1-e^{(1-u)})^{m+1}}{(m+1)^k} \\
&= \frac{1}{u-e^x}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^k}(e^{(1-u)}-1)^{m+1} \\
&= \frac{1}{u-e^x}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(m+1)!}{(m+1)^k}\frac{(e^{(1-u)}-1)^{m+1}}{(m+1)!} \\
&= \frac{1}{u-e^x}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(m+1)!}{(m+1)^k}\sum_{n=m+1}^{\infty} S(n,m+1)\frac{(1-u)^n}{n!} \\
&= \frac{1-u}{u-e^x}\sum_{n=1}^{\infty}\frac{(1-u)^{n-1}}{n!}\sum_{m=0}^{n-1}\frac{(-1)^{m+1}(m+1)!}{(m+1)^k}S(n,m+1).
\end{aligned}$$

Hence we have the following:

PROPOSITION 1.

$$H_n^{(k)}(u) = -H_n(u)\sum_{l=1}^{\infty}\frac{(1-u)^{l-1}}{l!}\sum_{m=0}^{l-1}\frac{(-1)^{m+1}(m+1)!}{(m+1)^k}S(l,m+1)$$

for $n \geq 0$, $\forall k \geq 1$.

COROLLARY 1. *For $k \geq 1$,*

$$Li_k(1-e^{(1-u)}) = \sum_{l=1}^{\infty}(-1)^l \frac{(u-1)^l}{l!} \sum_{m=0}^{l-1} \frac{(-1)^{m+1}(m+1)!}{(m+1)^k} S(l, m+1).$$

By (4), we see that

$$\begin{aligned} &\frac{Li_k(1-e^{(1-u)})}{u-e^t} e^{xt} \\ &= \frac{Li_k(1-e^{(1-u)})}{u} \sum_{a=0}^{f-1} \frac{u^{f-a}}{Li_k(1-e^{(1-u^f)})} e^{(\frac{x+a}{f})ft} \frac{Li_k(1-e^{(1-u^f)})}{u^f - e^{ft}} \\ &= \frac{Li_k(1-e^{(1-u)})}{u} \sum_{a=0}^{f-1} \frac{u^{f-a}}{Li_k(1-e^{(1-u^f)})} e^{H^{(k)}(u^f; \frac{x+a}{f})ft}. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{H_m^{(k)}(u;x)}{m!} t^m \\ &= \frac{Li_k(1-e^{(1-u)})}{u} \sum_{a=0}^{f-1} \frac{u^{f-a}}{Li_k(1-e^{(1-u^f)})} \sum_{m=0}^{\infty} \frac{H_m^{(k)}(u^f; \frac{x+a}{f})}{m!} f^m t^m. \end{aligned}$$

Therefore we obtain the following:

PROPOSITION 2. *For $f \geq 1$, we have*

$$\begin{aligned} &f^n \sum_{a=0}^{f-1} \frac{u^{f-a}}{Li_k(1-e^{(1-u^f)})} H_n^{(k)}(u^f; \frac{x+a}{f}) \\ &= \frac{u}{Li_k(1-e^{(1-u)})} H_n^{(k)}(u;x) \end{aligned}$$

for all $k \geq 1$.

The above proposition is important for the construction of the p-adic poly-Euler measure in the next section.

3. Properties of poly-Eulerian numbers in p-adic case

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$.

Let f be a fixed positive integer, and let p be a fixed prime number. We denote

$$X = \varprojlim_N \mathbb{Z}/fp^N\mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<fp \\ p\nmid a}} a + fp\mathbb{Z}_p, \tag{8}$$

$$a + fp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{fp^N}\},$$

where $0 \le a < fp^N$.

Without loss of generality, we may assume that $u \in \mathbb{C}_p$ satisfies $|1 - u^{fp^n}|_p \ge 1$.

REMARK. If $|1-u^{fp^n}|_p \ge 1$, then $|Li_k(1-e^{(1-u^{fp^n})})|_p \ge 1$ due to the basic property of a non-Archimedean field that $|x+y|_p = \max\{|x|_p, |y|_p\}$ if $|x|_p \ne |y|_p$ ([4], [9]).

Let $a \in \mathbb{Z}$ with $0 \le a \le fp^n - 1$, $n \ge 0$. Then p-adic poly-Euler measure is defined by

$$\begin{aligned} &E^{(k)}_{poly;u;m}(a + fp^n\mathbb{Z}_p) \\ &= \frac{u^{fp^n - a}}{Li_k(1 - e^{(1-u^{fp^n})})}(fp^n)^m H^{(k)}_m(u^{fp^n}; \frac{a}{fp^n}). \end{aligned} \tag{9}$$

PROPOSITION 3 (Poly-Euler measure). *For* $m \ge 0$, $k \ge 1$, $E^{(k)}_{poly;u;m}$ *is measure on* X.

Proof. At frist, we show that $E^{(k)}_{poly;u;m}$ is a distribution on X. For that, it suffices to check that ([4], p. 35)

$$\begin{aligned} &\sum_{i=0}^{p-1} E^{(k)}_{poly;u;m}(a + ifp^n + fp^{n+1}\mathbb{Z}_p) \\ &= E^{(k)}_{poly;u;m}(a + fp^n\mathbb{Z}_p). \end{aligned}$$

The left hand side is equal to

$$\begin{aligned}
&\sum_{i=0}^{p-1} \frac{u^{fp^{n+1}-a-ifp^n}}{Li_k(1-e^{(1-u^{fp^{n+1}})})}(fp^{n+1})^m H_m^{(k)}\left(u^{fp^{n+1}}; \frac{a+ifp^n}{fp^{n+1}}\right) \\
&= p^m \sum_{i=0}^{p-1} \frac{(u^{fp^n})^{p-i} u^{-a} (fp^n)^m}{Li_k(1-e^{(1-(u^{fp^n})^p)})} H_m^{(k)}\left((u^{fp^n})^p; \frac{\frac{a}{fp^n}+i}{p}\right) \\
&= u^{-a}(fp^n)^m \frac{u^{fp^n}}{Li_k(1-e^{(1-u^{fp^n})})} H_m^{(k)}\left(u^{fp^n}; \frac{a}{fp^n}\right) \\
&= E_{poly;u;m}^{(k)}(a+fp^n\mathbb{Z}_p).
\end{aligned}$$

Next, by definition of $H_m^{(k)}(u;x)$ and the condition $|Li_k(1-e^{(1-u^{fp^n})})|_p \geq 1$ it is easy to show that $|E_{poly;u;m}^{(k)}(a+fp^n\mathbb{Z}_p)|_p \leq C$ for some constant C. Since every compact-open set U is a finite disjoint union of intervals $a+fp^n\mathbb{Z}_p$, $E_{poly;u;m}^{(k)}(U)$ is measure. □

COROLLARY 2. *For $m \geq 0$ and $k=1$,*

$$E_{poly;u;m}^{(1)}(a+fp^n\mathbb{Z}_p) = \frac{u^{fp^n-a}}{u^{fp^n}-1}(fp^n)^m H_m\left(u^{fp^n}; \frac{a}{fp^n}\right)$$

is measure on X. This is another generalized p-adic Euler measures.

REMARK. We denote $E_{poly;u}$ as

$$E_{poly;u} := -E_{poly;u;0}^{(1)}. \tag{10}$$

Thus we obtain the p-adic Euler measure $E_{poly;u}$ for $f=1$ ([2], [6], [7]). This measure yields an integral

$$\int_{\mathbb{Z}_p} a^m E_{poly;u}(a) = \lim_{n\to\infty} \sum_{a=0}^{p^n-1} a^m \frac{u^{p^n-a}}{1-u^{p^n}},$$

and the formula above can be written as

$$\int_{\mathbb{Z}_p} a^m E_{poly;u}(a) = \frac{u}{1-u} H_m(u) \quad \text{for } m \geq 0.$$

DEFINITION 1. Let χ be a primitive Dirichlet character with conductor f. For $n \geq 0,\ k \geq 1$, we define generalized poly-Euler numbers attached with primitive character χ as follows:

$$H_{n;\chi}^{(k)}(u) = f^n \sum_{a=1}^{f} u^{f-a}\chi(a)H_n^{(k)}\left(u^f; \frac{a}{f}\right). \tag{11}$$

REMARK. An alternative description of the generalized poly-Euler number is

$$H_{n,\chi}^{(k)}(u) = \sum_{a=1}^{f}\sum_{l=0}^{n} a^n u^{f-a}\chi(a)\binom{n}{l}H_l^{(k)}(u^f)\left(\frac{f}{a}\right)^l.$$

In particular, if $\chi = 1$, then $H_{n,1}^{(k)}(u) = \sum_{l=0}^{n}\binom{n}{l}H_l^{(k)}(u)$.

We can express the poly-Euler numbers as an integral over X by using the measure $E_{poly;u;m}^{(k)}$, that is,

$$\begin{aligned}
&\int_X \chi(x)E_{poly;u;m}^{(k)}(x)\\
&= \lim_{n\to\infty}\sum_{a=0}^{fp^n-1}\chi(a)E_{poly;u;m}^{(k)}(a+fp^n\mathbb{Z}_p)\\
&= \lim_{n\to\infty}\sum_{a=0}^{fp^n-1}\chi(a)\frac{u^{fp^n-a}}{Li_k(1-e^{(1-u^{fp^n})})}(fp^n)^m H_m^{(k)}\left(u^{fp^n}; \frac{a}{fp^n}\right)\\
&= \lim_{n\to\infty} f^m\sum_{a=0}^{f-1}\chi(a)u^{-a}(p^n)^m\sum_{i=0}^{p^n-1}\frac{(u^f)^{p^n-i}}{Li_k(1-e^{(1-u^{fp^n})})}H_m^{(k)}\left(u^{fp^n}; \frac{\frac{a}{f}+i}{p^n}\right)\\
&= f^m\sum_{a=0}^{f-1}\chi(a)u^{-a}\frac{u^f}{Li_k(1-e^{(1-u^f)})}H_m^{(k)}\left(u^f; \frac{a}{f}\right)\\
&= \frac{f^m}{Li_k(1-e^{(1-u^f)})}\sum_{a=0}^{f-1}\chi(a)u^{f-a}H_m^{(k)}\left(u^f; \frac{a}{f}\right)\\
&= \frac{H_{m,\chi}^{(k)}(u)}{Li_k(1-e^{(1-u^f)})} \quad \text{if } \chi \neq 1.
\end{aligned}$$

We have the following:

PROPOSITION 4. *For $m \geq 1$,*

$$\int_X \chi(x) E^{(k)}_{poly;u;m}(x) = \begin{cases} \frac{H^{(k)}_{m,\chi}(u)}{Li_k(1-e^{(1-u^f)})} & \text{if } \chi \neq 1, \\ \frac{uH^{(k)}_m(u)}{Li_k(1-e^{(1-u)})} & \text{if } \chi = 1. \end{cases}$$

THEOREM 1. *Let $f : X \to X$ be the function $f(x) = x^m$, m is a fixed positive integer. Then for all compact open $U \subset X$,*

$$\int_U 1\, E^{(k)}_{poly;u;m}(x) = \int_U f\, E_{poly;u}(x),$$

where $E_{poly;u}$ be the p-adic Euler measure on X in (10).

Proof. From (4) and with $u = u^{fp^n}, x = \frac{a}{fp^n}$, we obtain

$$\begin{aligned} &H^{(k)}_m\left(u^{fp^n}; \frac{a}{fp^n}\right) \\ &= \sum_{l=0}^{m} \binom{m}{l} H^{(k)}_l(u^{fp^n}) \left(\frac{a}{fp^n}\right)^{m-l} \\ &= H^{(k)}_0(u^{fp^n}) \left(\frac{a}{fp^n}\right)^m + \left[mH^{(k)}_1(u^{fp^n}) \left(\frac{a}{fp^n}\right)^{m-1} + \cdots \right] \end{aligned}$$

and

$$(fp^n)^m H^{(k)}_m\left(u^{fp^n}; \frac{a}{fp^n}\right) \equiv a^m H^{(k)}_0\left(u^{fp^n}\right) \pmod{p^n}.$$

By using this equation and (5), (10), we have

$$\begin{aligned} E^{(k)}_{poly;u;m}(a + fp^n\mathbb{Z}_p) &\equiv \frac{u^{fp^n - a}}{Li_k(1 - e^{1-u^{fp^n}})} a^m H^{(k)}_0(u^{fp^n}) \pmod{p^n} \\ &\equiv a^m \frac{u^{fp^n - a}}{-(1 - u^{fp^n})} \pmod{p^n} \\ &\equiv a^m E_{poly;u}(a + fp^n\mathbb{Z}_p) \pmod{p^n}. \end{aligned}$$

Hence

$$\begin{aligned}\int_U 1E_{poly;u;m}^{(k)}(x) &= \lim_{n\to\infty}\sum_{a=0}^{fp^n-1} E_{poly;u;m}(a+fp^n\mathbb{Z}_p)\\ &\equiv \lim_{n\to\infty}\sum_{a=0}^{fp^n-1} a^m E_{poly;u}(a+fp^n\mathbb{Z}_p) \pmod{p^n}\\ &= \int_U fE_{poly;u}(x).\end{aligned}$$

Thus we have the assertion. □

Let ω denote the Teichmüller character mod p (if $p=2$, mod 4) ([8]). For $x \in X^*$, we set $\langle x\rangle = \frac{x}{\omega(x)}$. Note that $\langle x\rangle^s$ is defined by $\exp(s\log_p\langle x\rangle)$, for $|s|_p \le 1$ since $|\langle x\rangle - 1|_p < p^{-1/(p-1)}$.

We define an interpolation function for poly-Euler numbers.

Definition 2. For $s \in \mathbb{Z}_p$,

$$\ell_p^{(k)}(u;s,\chi) = \int_{X^*} \langle x\rangle^{-s}\chi(x)x^{-m}\, E_{poly;u;m}^{(k)}(x).$$

Theorem 2. *For $m \ge 0$, we have*

$$\ell_p^{(k)}(u;-m,\chi\omega^m) = \begin{cases} \frac{H_{m,\chi}^{(k)}(u)}{Li_k(1-e^{(1-u^f)})} - \frac{p^m\chi(p)}{Li_k(1-e^{(1-u^{fp})})}H_{m,\chi}^{(k)}(u^p) \\ \text{if } \chi \ne 1, \\ \frac{uH_m^{(k)}(u)}{Li_k(1-e^{(1-u)})} - \frac{p^m u^p}{Li_k(1-e^{(1-u^p)})}H_m^{(k)}(u^p) \\ \text{if } \chi = 1. \end{cases}$$

Proof. We show that

$$\begin{aligned}
&\ell_p^{(k)}(u;-m,\chi\omega^m)\\
&=\int_{X^*}\langle x\rangle^m\chi(x)\omega^m(x)x^{-m}E^{(k)}_{poly;u;m}(x)\\
&=\int_{X^*}\langle x\rangle^m\chi(x)\omega^m(x)E_{poly;u}(x)\\
&=\int_{X^*}\chi(x)x^m E_{poly;u}(x)\\
&=\int_{X}\chi(x)x^m E_{poly;u}(x)-\int_{pX}\chi(x)x^m E_{poly;u}(x)\\
&=\int_{X}\chi(x)x^m E_{poly;u}(x)-\int_{X}\chi(px)(px)^m E_{poly;u}(px)\\
&=\int_{X}\chi(x)E^{(k)}_{poly;u;m}(x)-\int_{X}p^m\chi(p)\chi(x)E^{(k)}_{poly;u^p;m}(x).
\end{aligned}$$

By Proposition 4, we get the result. □

4. Some properties of Euler numbers

In (1), the m-th Euler number $H_m(u)$ attached to an algebraic number $|u|<1$ is defined by the generating function on an indeterminate t

$$\frac{1-u}{e^t-u}=e^{H(u)t}=\sum_{m=0}^{\infty}\frac{H_m(u)}{m!}t^m.$$

Set

$$S_m(n;u)=\frac{1}{u}+\frac{2^m}{u^2}+\cdots+\frac{(n-1)^m}{u^{n-1}}=\sum_{k=1}^{n-1}\frac{k^m}{u^k}$$

for $u\neq 0,1$. We will prove that Euler numbers are connected with the sums of series of $S_m(n;u)$.

LEMMA 1. *For any rational integer $m,n\geq 1$*

$$\frac{(n+H(u))^m}{u^n}-H_m(u)=\frac{1-u}{u}S_m(n;u).$$

Proof. We consider the equation

$$\begin{aligned} u^{-n}e^{(n+H(u))t} - e^{H(u)t} &= e^{H(u)t}(u^{-n}e^{nt} - 1) \\ &= \frac{1-u}{u}\sum_{r=0}^{n-1}\left(\frac{1}{u}\right)^r \sum_{m=0}^{\infty}\frac{r^m}{m!}t^m \\ &= \frac{1-u^n}{u^n} + \sum_{m=1}^{\infty}\frac{1-u}{u}S_m(n;u)\frac{t^m}{m!}. \end{aligned}$$

On the other hand,

$$u^{-n}e^{(n+H(u))t} - e^{H(u)t} = \sum_{m=1}^{\infty}\left(\frac{(n+H(u))^m}{u^n} - H_m(u)\right)\frac{t^m}{m!} + \frac{1-u^n}{u^n}.$$

Hence

$$\sum_{m=1}^{\infty}\left(\frac{(n+H(u))^m}{u^n} - H_m(u)\right)\frac{t^m}{m!} = \sum_{m=1}^{\infty}\frac{1-u}{u}S_m(n;u)\frac{t^m}{m!},$$

which proves the lemma. □

By Lemma 1, for $k \geq 1$ we have

$$\sum_{x=1}^{m}\frac{x^k}{u^x} = \frac{u}{1-u}\left(\frac{(m+H(u))^k}{u^m} - H_k(u)\right) + \frac{m^k}{u^m}. \tag{12}$$

The equation (12) implies

$$\begin{aligned} \sum_{x=1}^{n}\sum_{y=1}^{x}\frac{y^k}{u^y} &= n\frac{1^k}{u^1} + (n-1)\frac{2^k}{u^2} + \cdots + 1\frac{n^k}{u^n} \\ &= \frac{u}{1-u}\Bigg((n+1)\left(\frac{(n+H(u))^k}{u^n} - H_k(u)\right) \\ &\quad - \left(\frac{(n+H(u))^{k+1}}{u^n} - H_{k+1}(u)\right)\Bigg) + \frac{n^k}{u^n}. \end{aligned} \tag{13}$$

Also we have

(14)

$$\sum_{x=1}^{n}\sum_{y=1}^{x}\frac{y^k}{u^y}=\sum_{x=1}^{n}\left(\frac{u}{1-u}\left(\frac{(x+H(u))^k}{u^x}-H_k(u)\right)+\frac{x^k}{u^x}\right)$$
$$=\left(\frac{u}{1-u}\right)^2\left(\frac{(n+H(u)+H(u))^k}{u^n}-(H(u)+H(u))^k\right)$$
$$+\frac{2u}{1-u}\frac{(n+H(u))^k}{u^n}-\frac{u}{1-u}(n+2)H_k(u)+\frac{n^k}{u^n}.$$

Using (13) and (14)

$$(15)\qquad \begin{aligned}&(n-1)\frac{(n+H(u))^k}{u^n}-\frac{(n+H(u))^{k+1}}{u^n}+H_k(u)+H_{k+1}(u)\\&=\frac{u}{1-u}\left(\frac{(n+H(u)+H(u))^k}{u^n}-(H(u)+H(u))^k\right).\end{aligned}$$

We consider the coefficient for n in (15). Then since $u^{-n}=\sum_{l=0}^{\infty}\frac{(-n)^l}{l!}(\log u)^l$,

$$(16)\qquad \begin{aligned}&-k(H_{k-1}(u)+H_k(u))+(H_k(u)+H_{k+1}(u))\log u\\&=\frac{u}{1-u}(k(H(u)+H(u))^{k-1}-(H(u)+H(u))^k\log u).\end{aligned}$$

Therefore we obtain the following theorem.

THEOREM 3. *For any rational integer $k\geq 1$*

$$\sum_{j=0}^{k-1}k\binom{k-1}{j}H_{k-j-1}(u)H_j(u)-\sum_{j=0}^{k}\log u\binom{k}{j}H_{k-j}(u)H_j(u)$$
$$=\frac{1-u}{u}(k(H_k(u)-H_{k-1}(u))+\log u(H_{k+1}(u)+H_k(u))).$$

REMARK. We know that the n-th Bernoulli number B_χ^n belonging to a Dirichlet character χ can be expressed with $(n-1)$-th Euler numbers ([6]).

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DEPARTMENT OF MATHEMATICS, KYUNGANAM UNIVERSITY, MASAN 631-701, KOREA
E-mail: sonjin@hanma.kyungnam.ac.kr