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CHARACTERIZATIONS OF IDEAL WEAKLY $\delta\theta$-REFINABLE SPACES

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ABSTRACT. In this paper, we are interested in studying weak covering properties in the presence of a countable compact condition. The purpose of this paper is to characterize an ideal weakly $\delta\theta$-refinable space and to show that every ideal weakly $\delta\theta$-refinable space is isocompact. Also, we consider the behavior under mappings of ideal weakly $\delta\theta$-refinable properties and productivity of ideal weakly $\delta\theta$-refinable properties.

1. Introduction

A space X is said to be *isocompact* ([2]) if every closed countably compact subset of X is compact. The most obvious example of isocompact spaces is a Lindelöf space. Among the classes of spaces having the isocompactness property are neighborhood $\mathcal{F}$-spaces ([8]), spaces satisfying property θL ([7]), weakly $[\omega_1, \infty)^r$-refinable spaces ([14]), $\delta\theta$-penetrable spaces ([4]), almost realcompact spaces ([9]), weakly $\delta\theta$-refinable spaces ([14]), weakly Borel complete spaces ([12]), and pure spaces ([1]).

In [12], Masami Sakai introduced a new large class of isocompact spaces, called "κ-neat spaces". This class contains all of the above mentioned classes. Moreover, he proved that every neat space is isocompact.

In [16], Wicke and Worrell defined a covering property, called star reducible, possessed by all $\delta\theta$-refinable countably subparacompact spaces

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(Remark 1.4 in [16]) and also introduced weak star reducibility which is obviously weaker than star reducibility. Recently, the author showed ([6]) that every ultrapure space is weakly star reducible, that every weakly $[\omega_1, \infty)^r$-refinable space is weakly star reducible, and that every weakly star reducible space is κ-neat for any cardinal κ. Thus a weakly star reducible space is another space which is in the class of isocompact spaces.

It turns out that there is much interplay between general covering properties described using ultrafilters and general covering properties characterized by maximal open ideals and generalized realcompactness properties described in [9].

The purpose of this paper is to characterize an ideal weakly $\delta\theta$-refinable space (Theorem 3.1, Theorem 3.2, and Theorem 3.8) and to show that every ideal weakly $\delta\theta$-refinable space is isocompact (Theorem 3.6). Also, we consider the behavior under mappings of ideal weakly $\delta\theta$-refinable properties (Theorem 3.9) and productivity of ideal weakly $\delta\theta$-refinable properties (Theorem 4.1, Theorem 4.2, and Theorem 4.5).

This paper is organized as follows: Section 1 is an introduction. Section 2 consists of preliminaries which involve definitions and basic implications of weak covering properties. Section 3 is devoted to characterizations of ideal weakly $\delta\theta$-refinable spaces which are main results in this paper. Section 4 consists of some interesting results related to products of ideal weakly $\delta\theta$-refinable spaces.

Throughout this paper, all spaces will be assumed to be T_1 and we use the following notation:

For any set $A \subset X$ and a collection $\mathcal{U}$ of subsets of X, $st(A,\mathcal{U})$ (the *star* of $\mathcal{U}$ about A) denotes the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

If $x \in A$, $st(\{x\},\mathcal{U})$ is simply denoted by $st(x,\mathcal{U})$. $ord(x,\mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$, (where $|E|$ denotes the cardinal of the set E), $[\mathcal{U}]^{<\omega} = \{\mathcal{K} \subset \mathcal{U} : \mathcal{K}$ is finite $\}$, and $[\mathcal{U}]^{\omega} = \{\mathcal{K} \subset \mathcal{U} : \mathcal{K}$ is countable $\}$.

Also, if $\mathcal{V}$ is a collection of subsets of X and $x \in X$, then $\mathcal{V}(x) = \{V \in \mathcal{V} : x \in V\}$ and $I(x,\mathcal{V}) = \bigcap \mathcal{V}(x)$.

If $\mathcal{V} = \{\mathcal{V}_\alpha\}$ is a family of collections of subsets of X, then we denote by $\bigcup \mathcal{V}_\alpha = \bigcup_{V \in \mathcal{V}_\alpha} V$ and $\bigcup\bigcup \mathcal{V} = \bigcup(\bigcup \mathcal{V}_\alpha)$.

2. Preliminaries

We establish some convenient terminology used throughout the rest of this paper. As far as topological concepts are concerned, we follow [5] and [10]. First, we give the definition of a closed filter and its dual concept, an open ideal.

DEFINITION 2.1. A collection $\mathcal{F}$ of closed subsets of a space X is called a closed filter if

(a) $\emptyset \notin \mathcal{F}$,

(b) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$,

(c) if F_2 is closed in X and $F_1 \subset F_2$, and $F_1 \in \mathcal{F}$, then $F_2 \in \mathcal{F}$.

DEFINITION 2.2. A collection $\mathcal{G}$ of open sets in a space X is called an open ideal if

(a) $X \notin \mathcal{G}$,

(b) If $U_1, U_2 \in \mathcal{G}$, then $U_1 \cup U_2 \in \mathcal{G}$,

(c) If U_1 is open in X and $U_1 \subset U_2 \in \mathcal{G}$, then $U_1 \in \mathcal{G}$.

An open ideal in a space X is a covering of X if $\bigcup \mathcal{G} = X$. Throughout this paper, we will use the abbreviation a *moc* ideal to denote a maximal open covering ideal. However, a moc ideal in a space is called a maximal open cover in some literatures (for example, [9]).

DEFINITION 2.3 ([3],[14]). A space X is said to be *weakly θ-refinable* (resp. *weakly $\delta\theta$-refinable*) if for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ of $\mathcal{U}$ such that if $x \in X$ there is some $n \in \omega$ with $0 < ord(x, \mathcal{G}_n) < \omega$ (resp. $0 < ord(x, \mathcal{G}_n) \leq \omega$).

Moreover, if each $\mathcal{G}_n$ covers X, then X is said to be *θ-refinable* (resp. *$\delta\theta$-refinable* or *submeta-Lindelöf*).

DEFINITION 2.4 ([1]). A countable family $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of collections of subsets of a space X is called an *interlacing* on X if $\bigcup\bigcup \mathcal{V} = X$ and for each $n \in \omega$, each $V \in \mathcal{V}_n$ is open in $\bigcup \mathcal{V}_n$.

An interlacing $\mathcal{V}$ is called *suspended* (resp. *δ-suspended*) from a family $\mathcal{H}$ of subsets of a space X if for every $n \in \omega$ and $x \in \bigcup \mathcal{V}_n$, there is a finite family $\mathcal{K} \in [\mathcal{H}]^{<\omega}$ (resp. a countable family $\mathcal{K} \in [\mathcal{H}]^{\omega}$) such that $st(x, \mathcal{V}_n) \cap (\bigcap \mathcal{K}) = \emptyset$.

A space X is called *ultrapure* if for each free closed collection $\mathcal{F}$ on X there is an interlacing which is δ-suspended from $\mathcal{F}$.

A space X is called *astral* if for every countably prime free closed filter $\mathcal{F}$ on X with c.i.p. there exists an interlacing which is δ-suspended from $\mathcal{F}$.

A space X is called *pure* if for each free closed ultrafilter $\mathcal{F}$ on X there is an interlacing which is δ-suspended from $\mathcal{F}$.

Note that ultrapure implies astral implies pure. It is known that in the case of ultrafilters with the countable intersection property (c.i.p.), the terms suspended and δ-suspended coincide.

The following theorem is due to Arhangel'skiĭ.

THEOREM 2.5. *Every weakly $\delta\theta$-refinable space X is ultrapure.*

Proof. See ([1]). □

The following definition is a covering property which is weaker than weakly $\delta\theta$-refinable.

DEFINITION 2.6 ([3]). An open cover $\mathcal{G} = \bigcup\{\mathcal{G}_n : n \in \omega\}$ of a space X is a θ-penetration (resp. $\delta\theta$-penetration) of a cover $\mathcal{U}$ of X if for every $x \in X$, $\bigcap\{I(x, \mathcal{G}_n) : n \in \omega \text{ and } 0 < ord(x, \mathcal{G}_n) < \omega\} \subset U$ for some $U \in \mathcal{U}$ (resp. $\bigcap\{I(x, \mathcal{G}_n) : n \in \omega \text{ and } 0 < ord(x, \mathcal{G}_n) \leq \omega\} \subset U$ for some $U \in \mathcal{U}$). A space X is θ-penetrable (resp. $\delta\theta$-penetrable) if every open cover of X has a θ-penetration (resp. $\delta\theta$-penetration).

In fact, it is easy to check that every weak θ-refinement (resp. weak $\delta\theta$-refinement) of $\mathcal{U}$ is a θ-penetration (resp. $\delta\theta$-penetration) of $\mathcal{U}$. However, it is known [4, Remark 2.1] that the converse is not true in general.

In [14], if X is countably compact and weakly $[\omega_1, \infty)^r$-refinable, then X is compact. This says that weakly $[\omega_1, \infty)^r$-refinable spaces are isocompact.

There are other weak covering properties which imply isocompactness. For example, Davis in [7] studied 'property θL' and showed that this property generalizes weak $\delta\theta$-refinability and implies isocompactness. For other conditions which force a countably compact space to be compact, see [13].

THEOREM 2.7 ([1]). *Every countably compact, pure space is compact.*

Proof. See [13]. □

REMARK. Theorem 2.7 shows that every pure space is isocompact.

For a cardinal κ, the *cofinality* of κ, denoted by $cf(\kappa)$, is the smallest cardinal λ such that κ has a cofinal subset of cadinality λ. A cardinal κ is *regular* if $\kappa \geq \omega$ and $cf(\kappa) = \kappa$.

DEFINITION 2.8 ([16]). A cover $\mathcal{U}$ of a space X is called *regularly rigid* if no subcollection of $\mathcal{U}$ of cardinality less than $|\mathcal{U}|$ covers X and $|\mathcal{U}|$ is regular or $1 < |\mathcal{U}| < \omega$.

DEFINITION 2.9 ([16]). A space X is called *star reducible* if for every regularly rigid open cover $\mathcal{H}$ of X, there exists a sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of open covers of X such that for all $p \in X$ there exist $n \in \omega$ and $\mathcal{H}' \subset \mathcal{H}$ such that $|\mathcal{H}'| < |\mathcal{H}|$ and $\mathcal{H}'$ covers $st(p, \mathcal{G}_n)$.

DEFINITION 2.10 ([6]). A space X is called *weakly star reducible* if for every uncountable regularly rigid open cover $\mathcal{U}$ of X there is a collection $\mathcal{V}$ of collections of subsets of X such that:
(i) $\bigcup\bigcup\mathcal{V} = X$,
(ii) $|\mathcal{V}| < |\mathcal{U}|$,
(iii) for all $\mathcal{G} \in \mathcal{V}$ and for all $G \in \mathcal{G}$, G is open in $\bigcup\mathcal{G}$, and
(iv) for all $p \in X$, there exist $\mathcal{G} \in \mathcal{V}$ and $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| < |\mathcal{U}|$ and $st(p, \mathcal{G}) \subset \bigcup\mathcal{U}'$.

The above definition is essentially based on Definition 4.8 in [16]. Clearly, weak star reducibility is obviously weaker than star reducibility and every developable space is star reducible (Remark 1.5 in [16]) and thus weakly star reducible.

Define for each free closed ultrafilter $\mathcal{H}$ on X with c.i.p., $\lambda(\mathcal{H}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{H}, \bigcap\mathcal{F} = \emptyset\}$. Note that $\lambda(\mathcal{H})$ is an uncountable regular cardinal.

DEFINITION 2.11 ([12]). Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p. and κ be a cardinal number. A system $\langle \{X_\gamma\}, \{\mathcal{V}_\gamma\}, \{f_\gamma\} \rangle_{\gamma\in\Gamma}$ is called a *κ-neat system* for $\mathcal{H}$ if the following are satisfied:

(1) $|\Gamma| < \lambda(\mathcal{H})$.

(2) $\{X_\gamma\}_{\gamma\in\Gamma}$ is a cover of X and $\mathcal{V}_\gamma$ is an open collection of X such that $X_\gamma \subset \bigcup \mathcal{V}_\gamma$ for each $\gamma \in \Gamma$.
(3) Each f_γ is a function from X_γ to $\mathcal{V}_\gamma$ such that if $A \subset X_\gamma$, $|A| \leq \kappa$ and $f_\gamma|A$ is injective, then the closure of A in $\bigcup \mathcal{V}_\gamma$ is contained in $\bigcup_{x\in A} f_\gamma(x)$.
(4) For each $\gamma \in \Gamma$ and $x \in X_\gamma$ there exists $H \in \mathcal{H}$ such that $f_\gamma(x) \cap X_\gamma \cap H = \emptyset$.

A space X is called a *κ-neat space* if for each free closed ultrafilter $\mathcal{H}$ on X with c.i.p. there exists a κ-neat system for $\mathcal{H}$. An ω-neat space is merely called a *neat space.*

THEOREM 2.12 (Theorem 2.6 in [12]). *Every neat space is isocompact.*

3. Characterizations

If $\mathcal{U}$ and $\mathcal{V}$ are covers of a set X, the collection $\mathcal{V}$ is said to be a *refinement* of the collection $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ with $V \subseteq U$. If the collection $\mathcal{V}$ above is not required to cover X, we say that $\mathcal{V}$ is a *partial refinement* of $\mathcal{U}$.

A space X is called *ideal weakly θ-refinable* (*resp. ideal weakly $\delta\theta$-refinable*) if for every moc ideal $\mathcal{U}$ there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ of open partial refinements of $\mathcal{U}$ such that for each $x \in X$ there is $n \in \omega$ with $0 < ord(x, \mathcal{V}_n) < \omega$ (resp. $n \in \omega$ with $0 < ord(x, \mathcal{V}_n) \leq \omega$). Clearly, every ideal weakly θ-refinable space is ideal weakly $\delta\theta$-refinable. We give some characterizations of ideal weak $\delta\theta$-refinability with the interlacing terminology introduced by Arhangelskii ([1]).

THEOREM 3.1. *A space X is ideal weakly $\delta\theta$-refinable if and only if for every moc ideal $\mathcal{U}$ of X there exists a point-countable interlacing $\mathcal{V}$ such that $\bigcup \mathcal{V}$ refines $\mathcal{U}$.*

Proof. ($\Rightarrow$) Suppose that X is ideal weakly $\delta\theta$-refinable. Let $\mathcal{U}$ be a moc ideal of X. Then there exists a weak $\delta\theta$-refinement $\mathcal{V}$ of $\mathcal{U}$, i.e., there exists a sequence $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of open partial refinements of $\mathcal{U}$ such that for each $x \in X$, there exists $n \in \omega$ with $0 < ord(x, \mathcal{V}_n) \leq \omega$.

For each $n \in \omega$, let

$$X_n = \{x \in \bigcup \mathcal{V}_n : n = \min\{k : x \in \bigcup \mathcal{V}_k \text{ and } 0 < ord(x, \mathcal{V}_k) \leq \omega\}\}.$$

and let $\mathcal{V}'_n = \{V \cap X_n : V \in \mathcal{V}_n\}$. Then for each $n \in \omega$ and $V' \in \mathcal{V}'_n$, $V' = V \cap X_n = V \cap \bigcup \mathcal{V}'_n$ for some $V \in \mathcal{V}_n$. Also $\bigcup \bigcup \mathcal{V}'_n = X$ since for every $x \in X$ there exists $n \in \omega$ such that $0 < ord(x, \mathcal{V}_n) \leq \omega$. So $\mathcal{V}' = \{\mathcal{V}'_n : n \in \omega\}$ is an interlacing and it is clearly point-countable, and $\bigcup \mathcal{V}'$ refines $\mathcal{U}$.

($\Leftarrow$) Suppose the condition holds. Let $\mathcal{U}$ be a moc ideal of X. Then by assumption there exists a point-countable interlacing $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ whose union refines $\mathcal{U}$. For each $n \in \omega$ and $V \in \mathcal{V}_n$ choose an open set V' in X such that $V' \subset U$ for some $U \in \mathcal{U}$ and $V = V' \cap (\bigcup \mathcal{V}_n)$. Let $\mathcal{V}'_n = \{V' : V \in \mathcal{V}_n\}$ for all $n \in \omega$. Then $\{\mathcal{V}'_n : n \in \omega\}$ is a sequence of open partial refinements of $\mathcal{U}$. Moreover, if $x \in X$, then there exists $n \in \omega$ with $x \in \bigcup \mathcal{V}_n$ because $\mathcal{V}$ is an interlacing. Consequently, $x \in \bigcup \mathcal{V}'_n$ and $0 < ord(x, \mathcal{V}'_n) \leq \omega$ since $\mathcal{V}$ is point-countable. Therefore X is ideal weakly $\delta\theta$-refinable. □

THEOREM 3.2. *A space X is ideal weakly $\delta\theta$-refinable if and only if for every closed ultrafilter $\mathcal{F}$ on X there exists a point-countable interlacing $\mathcal{V}$ which is δ-suspended from $\mathcal{F}$.*

Proof. It follows directly from definitions of open ideals and closed ultrafilters. □

COROLLARY 3.3. *Every ideal weakly $\delta\theta$-refinable (or even ideal weakly θ-refinable) space is pure.*

Now we consider some of basic properties of ideal weakly $\delta\theta$-refinable spaces which will be in the class of spaces having the isocompactness property.

PROPOSITION 3.4. *Every countably compact ideal weakly $\delta\theta$-refinable space is compact.*

Proof. Let X be a countably compact ideal weakly $\delta\theta$-refinable space. Then it follows from Corollary 3.3 that X is pure. So by Theorem 2.7, X is compact. □

We will show that ideal weak $\delta\theta$-refinability is hereditary with respect to closed subsets.

PROPOSITION 3.5. *Every closed subset of an ideal weakly $\delta\theta$-refinable space is ideal weakly $\delta\theta$-refinable.*

Proof. Let X be an ideal weakly $\delta\theta$-refinable space and $C \subset X$ be closed. To apply Theorem 3.2, let $\mathcal{F}$ be a closed ultrafilter on C and let $\mathcal{H}$ be a closed ultrafilter on X which contains $\mathcal{F}$. Then by Theorem 3.2, there is a point-countable interlacing $\mathcal{E}' = \{\mathcal{E}'_n : n \in \omega\}$ on X which is δ-suspended from $\mathcal{H}$. So if we let $\mathcal{E}_n = \{E \cap C : E \in \mathcal{E}'_n\}$ and $\mathcal{E} = \{\mathcal{E}_n : n \in \omega\}$, then it is easy to see that $\mathcal{E}$ is the required interlacing. Thus, by Theorem 3.2, C is ideal weakly $\delta\theta$-refinable. □

THEOREM 3.6. *Every ideal weakly $\delta\theta$-refinable space is isocompact.*

Proof. It follows from Proposition 3.4 and Proposition 3.5. □

Even though ideal weakly $\delta\theta$-refinable spaces and weakly star reducible spaces are isocompact (see [6]), the author does not know whether every ideal weakly $\delta\theta$-refinable space is weakly star reducible.

THEOREM 3.7 (LEMMA 2.2 [4]). *A space X is ideal weakly θ-refinable if and only if for every moc ideal $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ which is a θ-penetration of $\mathcal{U}$.*

The following theorem is an analog of Theorem 3.7 for ideal weak $\delta\theta$-refinability.

THEOREM 3.8. *A space X is ideal weakly $\delta\theta$-refinable if and only if for every moc ideal $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ which is a $\delta\theta$-penetration of $\mathcal{U}$.*

Proof. ($\Rightarrow$) Suppose that X is an ideal weakly $\delta\theta$-refinable space. Let $\mathcal{U}$ be a moc ideal of X. Then there exists a partial refinement $\{\mathcal{V}_n : n \in \omega\}$ of $\mathcal{U}$ such that for every $x \in X$ there exists $n \in \omega$ such that $0 < ord(x, \mathcal{V}_n) \leq \omega$. So $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$ is a $\delta\theta$-penetration of $\mathcal{U}$.

($\Leftarrow$) Suppose the condition holds. Let $\mathcal{U}$ be a moc ideal of X. Without loss of generality, we may assume that $\mathcal{U}$ is closed under countable unions; in fact, if not, then there exists a countable subcollection $\{G_n : n \in \omega\}$ of $\mathcal{U}$ such that $\bigcup_{n\in\omega} G_n \notin \mathcal{U}$. By maximality of $\mathcal{U}$ with $X \notin \mathcal{U}$, there exists a countable subcover $\{G'_n : n \in \omega\}$ of $\mathcal{U}$. Then $\{\{G'_n\} : n \in \omega\}$ is a $\delta\theta$-refinement of $\mathcal{U}$.

Now let $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$ be a $\delta\theta$-penetration of $\mathcal{U}$. For each $n \in \omega$, we define

$$X_n = \{x \in X : 0 < ord(x, \mathcal{V}_n) \leq \omega \,\text{and}\, V \in \mathcal{U} \,\text{for some}\, V \in \mathcal{V}_n(x)\}.$$

Also, for each $x \in X$, we define

$$f(x) = \{n \in \omega : 0 < ord(x, \mathcal{V}_n) \leq \omega\}.$$

We claim that $X = \bigcup_{n\in\omega} X_n$. Suppose for the contradiction that that there is a $y \in X$ with $y \notin \bigcup_{n\in\omega} X_n$. Let $\mathcal{G} = \{(n, V) : n \in f(y) \,\text{and}\, V \in \mathcal{V}_n(y)\}$. Then for each pair $(n, V) \in \mathcal{G}$, we have $0 < ord(x, \mathcal{V}_n) \leq \omega$, $V \in \mathcal{V}_n(y)$, and $y \notin X_n$. Thus $V \notin \mathcal{U}$. Hence there is a $G(n, V) \in \mathcal{U}$ such that $V \bigcup G(n, V) = X$, i.e., $X \backslash V \subset G(n, V)$. Now since $\mathcal{U}$ is closed under countable unions, we have

$$\bigcup_{n\in f(y)} \bigcup\{G(n, V) : V \in \mathcal{V}_n(y)\} \in \mathcal{U}.$$

Since $\bigcap\{I(y, \mathcal{V}_n) : n \in f(y)\} \subset U$ for some $U \in \mathcal{U}$ by $\delta\theta$-penetrability, we have

$$\begin{aligned} X =& [X \backslash \bigcap\{I(y, \mathcal{V}_n) : n \in f(y)\}] \cup U \\ =& \bigcup\{X \backslash I(y, \mathcal{V}_n) : n \in f(y)\} \cup U \\ =& \bigcup_{n\in f(y)} \bigcup\{X \backslash V : V \in \mathcal{V}_n(y)\} \cup U \\ \subset& \bigcup_{n\in f(y)} \bigcup\{G(n, V) : V \in \mathcal{V}_n(y)\} \cup U. \end{aligned}$$

This gives us that the union of two elements of $\mathcal{U}$ equals to X. So we have a contradiction because $\mathcal{U}$ is an ideal. This proves our claim.

Now for all $n \in \omega$ and $x \in X_n$, there exists a $V(x,n) \in \mathcal{V}_n(x)$ with $V(x,n) \in \mathcal{U}$. For each $n \in \omega$, let $\mathcal{V}'_n = \{V(x,n) : x \in X_n\}$ and let $\mathcal{V}' = \{\mathcal{V}'_n : n \in \omega\}$. Finally, we claim that $\mathcal{V}'$ is a weak $\delta\theta$-refinement of $\mathcal{U}$. Let $x \in X$. Then $x \in X_n$ for some $n \in \omega$. Then $x \in V(x,n) \in \mathcal{V}'_n$ and $|\mathcal{V}'_n(x)| \leq |\mathcal{V}_n(x)| \leq \omega$. This completes the proof. □

Throughout the rest of this section, we consider the behavior under mappings of ideal weakly $\delta\theta$-refinable properties.

THEOREM 3.9. *Let $f : X \to Y$ be a continuous mapping from a space X onto an ideal weakly $\delta\theta$-refinable space Y. Then every closed inverse image under f is ideal weakly $\delta\theta$-refinable in X if and only if each fiber $f^{-1}(y)$ is ideal weakly $\delta\theta$-refinable.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) We will apply Theorem 3.2 in this proof. To do this, let $\mathcal{F}$ be a free closed ultrafilter on X. Then $\mathcal{F}^{\#} = \{f(F) : F \in \mathcal{F}\}$ is a closed ultrafilter on Y. There are two possible cases as follows:

If $\mathcal{F}^{\#}$ is fixed, then there exist $y \in Y$ and $F' \in \mathcal{F}$ with $\bigcap \mathcal{F}^{\#} = \{y\}$ and $F' = f^{-1}(\{y\})$. So $\mathcal{F}_{F'} = \{F \cap F' : F \in \mathcal{F}\}$ is a free closed ultrafilter on F'. Thus there exists a point-countable interlacing $\mathcal{V}' = \{\mathcal{V}_n : n \in \omega\backslash\{0\}\}$ which is δ-suspended from $\mathcal{F}_{F'}$. Let $\mathcal{V}_0 = \{X\backslash F'\}$. Then $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ is a point-countable interlacing which is δ-suspended from $\mathcal{F}$.

If $\mathcal{F}^{\#}$ is free, then there is a point-countable interlacing $\mathcal{W} = \{\mathcal{W}_n : n \in \omega\}$ which is δ-suspended from $\mathcal{F}_{F'}$. For each $n \in \omega$, let $\mathcal{Z}_n = \{f^{-1}(W) : W \in \mathcal{W}_n\}$. Then $\mathcal{Z} = \{\mathcal{Z}_n : n \in \omega\}$ is the required point-countable interlacing δ-suspended from $\mathcal{F}$. We need only check the δ-suspension condition. Let $n \in \omega$ and $x \in \bigcup \mathcal{Z}_n$. Then there exists a countable $\mathcal{H}^{\#} \subset \mathcal{F}^{\#}$ with $st(f(x), \mathcal{W}_n) \cap (\bigcap \mathcal{H}^{\#}) = \emptyset$. If $\mathcal{H} = \{f^{-1}(H) : H \in \mathcal{H}^{\#}\}$, then $st(x, \mathcal{Z}_n) \cap (\bigcap \mathcal{H}) = f^{-1}(st(f(x), \mathcal{W}_n)) \cap f^{-1}(\bigcap \mathcal{H}^{\#}) = f^{-1}(\emptyset) = \emptyset$. □

COROLLARY 3.10. *The perfect inverse image of an ideal weakly $\delta\theta$-refinable space is ideal weakly $\delta\theta$-refinable.*

COROLLARY 3.11. *The topological sum of ideal weakly $\delta\theta$-refinable spaces is ideal weakly $\delta\theta$-refinable.*

Proof. Let $X = \bigoplus_{i \in I} X_i$, where I has the discrete topology. Then I is an ideal weakly $\delta\theta$-refinable space. Let $f : X \to I$ be defined by $f(x) = i$ if $x \in X_i$. Then f is a closed continuous surjective map with ideal weakly $\delta\theta$-refinable fibers and thus X is ideal weakly $\delta\theta$-refinable. □

4. Products

Tychonoff's theorem says that the product of compact spaces is compact. Unlike compactness, most covering properties are not productive, for example, the product space of paracompact spaces need not be paracompact. So we are interested in finding the conditions which make it possible for a product of topological spaces to inherit a special property from its factor spaces.

THEOREM 4.1. *The product of weakly $\delta\theta$-refinable spaces is ideal weakly $\delta\theta$-refinable.*

Proof. Let $X = \prod_{\alpha \in A} X_\alpha$ be the product space of weakly $\delta\theta$-refinable spaces X_α for each $\alpha \in A$, and let $\pi_\alpha : X \to X_\alpha$ be the α-th projection map for each $\alpha \in A$, and let $\mathcal{F}$ be a free closed ultrafilter on X. Then it is well known that there is a $\beta \in A$ such that

$$\mathcal{F}_\beta = \{F \subseteq X_\beta : F \text{ is closed in } X_\beta \text{ and } \pi_\beta^{-1}(F) \in \mathcal{F}\}$$

is a free prime filter on X_β. Let $\mathcal{E}_\beta$ be a point-countable interlacing on X_β which is δ-suspended from $\mathcal{F}_\beta$. Such an interlacing exists because weak $\delta\theta$-refinablity implies astral. For each $n \in \omega$, let $\mathcal{E}_n = \{\pi_\beta^{-1}(E) : E \in \mathcal{E}_{\beta,n}\}$. We claim that $\mathcal{E} = \{\mathcal{E}_n : n \in \omega\}$ is a point-countable interlacing on X which is δ-suspended from $\mathcal{F}$. Continuity of the projection map gives that $\mathcal{E}$ is an interlacing and the point-countable condition is clear. It remains to check that the δ-suspended condition. If $\pi_\beta^{-1}(E_\beta) \in \mathcal{E}_n$, then there exists $F_\beta \in \mathcal{F}_\beta$ with $F_\beta \cap E_\beta = \emptyset$, and so $\pi_\beta^{-1}(F_\beta) \cap \pi_\beta^{-1}(E_\beta) = \emptyset$. By the definition of $\mathcal{F}_\beta$, we get $\pi_\beta^{-1}(F_\beta) \in \mathcal{F}$, so $\mathcal{E}$ is δ-suspended from $\mathcal{F}$. □

THEOREM 4.2. *Let X be an ideal weakly $\delta\theta$-refinable space and Y be a compact space. Then the product space $X \times Y$ is ideal weakly $\delta\theta$-refinable.*

Proof. Let X and Y be such spaces. Then the projection map $\pi_X : X \times Y \to X$ is perfect since Y is compact. Thus by Corollary 3.10, $X \times Y$ is ideal weakly $\delta\theta$-refinable. □

In the rest of this paper, we consider products of ideal type properties with Borel complete spaces. Recall that a space X is called *Borel complete* if each ultrafilter of Borel sets with c.i.p. is fixed.

LEMMA 4.3 [11]. *Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p. If B is a Borel set of X, and if B contains no member of $\mathcal{H}$, then there is an $H \in \mathcal{H}$ such that $H \cap B = \emptyset$.*

LEMMA 4.4. *Let $f : X \to Y$ be continuous, $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p. Then*

$$\mathcal{B} = \{B \subseteq Y : B \text{ is a Borel set and } H \subseteq f^{-1}(B) \text{ for some } H \in \mathcal{H}\}$$

is a Borel ultrafilter on Y with c.i.p.

Proof. Clearly $\mathcal{B}$ has the c.i.p. Let $B \subseteq Y$ be a Borel set and $B \cap B' \neq \emptyset$ for each $B' \in \mathcal{B}$. If $B \notin \mathcal{B}$, then $f^{-1}(B)$ contains no member of $\mathcal{H}$, and $f^{-1}(B)$ is a Borel set because f is continuous and B is a Borel set. So by Lemma 4.3 there exists $H \in \mathcal{H}$ such that $H \cap f^{-1}(B) = \emptyset$. Then $f^{-1}(Y \backslash B) \supseteq H$. So $Y \backslash B \in \mathcal{B}$ and thus $B \cap (Y \backslash B) \neq \emptyset$, which is a contradiction. This proves that $\mathcal{B}$ is an ultrafilter. So the conclusion follows. □

THEOREM 4.5. *The product of a Borel complete space X and an ideal weakly $\delta\theta$-refinable space Y is ideal weakly $\delta\theta$-refinable.*

Proof. Let $\pi_X : X \times Y \to X$ be the projection map and $\mathcal{H}$ be a free closed ultrafilter on $X \times Y$. Without loss of generality we may assume that $\mathcal{H}$ has c.i.p. Let

$$\mathcal{B} = \{B \subseteq X : B \text{ is a Borel set and } H \subseteq \pi_X^{-1}(B) \text{ for some } H \in \mathcal{H}\}.$$

Then by Lemma 4.4, $\mathcal{B}$ is a Borel ultrafilter with c.i.p. on X. Therefore $\mathcal{B}$ is fixed, i.e., $\bigcap\mathcal{B} \neq \emptyset$, and so we have that $\pi_X^{-1}(\bigcap\mathcal{B}) \in \mathcal{H}$. Since it is in the form of $\{x\} \times Y$, it is ideal weakly $\delta\theta$-refinable; $\{x\} \times Y$ is a closed subspace of $X \times Y$ and an element of $\mathcal{H}$. Consequently, the proof follows from Theorem 3.9. □

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