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WEIGHTED WEAK TYPE ESTIMATES FOR CERTAIN MAXIMAL OPERATORS IN SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. Let ν be a positive Borel measure on a space of homogeneous type (X, d, μ), satisfying the doubling property. A condition on a weight w for which a maximal operator $M_\nu f(x)$ defined by

$$M_\nu f(x) = \sup_{r>0} \frac{1}{\nu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y),$$

is of weak type (p,p) with respect to (ν, w), is that there exists a constant C such that $C \leq w(y)$ for a.e. $y \in B(x,r)$ if $p = 1$, and $\left(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(y)^{-\frac{1}{p-1}} d\mu(y)\right)^{p-1} \leq C$, if $1 < p < \infty$.

1. Introduction

Let ν be a positive Borel measure on a space of homogeneous type (X, d, μ), satisfying that for a ball B $\nu(B) = 0$ implies $\mu(B) = 0$. Let M_ν be a maximal operator given by

$$M_\nu f(x) = \sup_{\delta>0} \frac{1}{\nu(B(x,\delta))} \int_{B(x,\delta)} |f(y)| d\mu(y), \tag{1.1}$$

where $B(x,\delta) = \{y \in X : d(x,y) < \delta\}$. Then it is well known that this operator satisfies a certain weak type property in the sense that

$$\nu(\{x : M_\nu f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| d\mu(y). \tag{1.2}$$

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When $\nu = \mu$, note that this maximal operator is the standard Hardy-Littlewood maximal operator. For M_ν refer to [1, 3, 4, 5]. Note that this is a centered maximal operator. For a noncentered case with $X = \mathbb{R}^n$ and Euclidean balls used, then (1.2) is still true for the case $n = 1$ but not for the case $n > 2$. For details, see [6].

In this paper we are going to obtain a necessary and sufficient condition for which $M_\nu f$ is of weak type (p, p) with respect to a given weight w which is defined by a nonnegative measurable functions on X. More precisely,

$$\nu(\{x : M_\nu f(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_X |f(y)|^p w(y) d\mu(y).$$

But as mentioned above, (1.2) does not hold in general case, so additional assumptions need to be imposed. In this paper we suppose that ν satisfies the doubling property:

$$0 < \nu(B(x, 2r)) \leq C\nu(B(x, r)) < \infty$$

some constant C.

DEFINITION 1.1. Let X be a topological space. Assume d is a pseudo-distance on X, i.e., a nonnegative function defined on $X \times X$ satisfying

(i) $d(x, x) = 0$; $d(x, y) > 0$ if $x \neq y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq K[d(x, y) + d(y, z)]$, where K is some fixed constant.
Assume further that
(iv) the balls $B(x, r) = \{y \in X : d(x, y) < r\}$ form a basis of open neighborhoods at $x \in X$ and that μ is a Borel measure on X such that
(v) $0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty$, where A is some fixed constant.
Then (X, d, μ) is called a space of homogeneous type.

REMARK. Properties (iii) and (v) will be referred to as the *triangle inequality* and the *doubling property.* respectively.

Note that the condition (b) is equivalent that for every $c > 0$, there exists a constant A_c such that $\mu(B(x, cr)) \leq A_c\mu(B(x, r))$.

Throughout the paper, (X, d, μ) denotes a space of homogeneous type.

DEFINITION 1.2. Let w be a nonnegative measurable functions on X. M_ν is of weak type (p,p) with respect to w if there exists a constant C, independent of f, such that

$$\nu(\{x : M_\nu f(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_X |f(y)|^p w(y) d\mu(y). \tag{1.3}$$

Throughout the paper, the constant C can be different at every line and may depend on A, K appearing in the definition of spaces of homogeneous type, but it is independent of particular functions.

2. Main theorems

The following lemma, called Vitali-Wiener type Covering Lemma, is the main tool. For the proof, refer to [2].

LEMMA 2.1. *Let E be a bounded subset of X, i.e., E is contained in some ball. Let $r(x)$ be a positive number for each $x \in E$. Then there is a (finite or infinite) sequence of disjoint balls $B(x_i, r(x_i))$, $x_i \in E$, such that the balls $B(x_i, 4Kr(x_i))$ cover E, where K is the constant in the triangle inequality. Furthermore, every $x \in E$ is contained in some ball $B(x_i, 4Kr(x_i))$ satisfying $r(x) \leq 2r(x_i)$.*

THEOREM 2.1. *Let w be a weight and $1 \leq p < \infty$. Assume that ν satisfies the doubling property. Then M_ν is of weak type (p,p) if and only if there exists a constant C such that*

$$C \leq w(y) \tag{2.1}$$

for μ-a.e. $c \in B(x,r)$ if $p = 1$, and

$$\left(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(y)^{-\frac{1}{p-1}} d\mu(y) \right)^{p-1} \leq C, \tag{2.2}$$

if $1 < p < \infty$.

Proof. Suppose that (1.3) holds for a given p with $1 \le p < \infty$. Let f be nonnegative measurable function on X. For a ball $B(x,r)$, write

$$f_{B(x,r)} = \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(y)d\mu(y)$$

and for each $\lambda > 0$,

$$E_\lambda = \{x \in \mathbb{R}^n : M_\nu f(x) > \lambda\}.$$

If $y \in B(x,r)$, then $B(x,r) \subset B(y,2Kr)$ and so

$$\begin{aligned} f_{B(x,r)} &\le \frac{\nu(B(y,2Kr))}{\nu(B(x,r))} \frac{1}{\nu(B(y,2Kr))} \int_{B(y,2Kr)} f(z)d\mu(z) \\ &\le CM_\nu f(y) \end{aligned} \tag{2.3}$$

for every $y \in B(x,r)$. Hence for any λ with $0 < \lambda < f_{B(x,r)}$, we have

$$B(x,r) \subset \{M_\nu f > \lambda/C\}$$

and so

$$\nu(B(x,r)) \le \nu(\{M_\nu f > \lambda/C\}) \le \frac{C}{\lambda^p} \int_X f(y)^p w(y)d\mu(y).$$

Therefore

$$f^p_{B(x,r)} \nu(B(x,r)) \le C \int_X f(y)^p w(y)d\mu(y). \tag{2.4}$$

Let S be a measurable subset of $B(x,r)$. Replacing f by $f\chi_S$ in (2.4), we obtain

$$\left(\frac{1}{\nu(B(x,r))} \int_S f(y)d\mu(y)\right)^p \nu(B(x,r)) \le C \int_S f(y)^p w(y)d\mu(y). \tag{2.5}$$

Suppose first that $p = 1$. Take $f \equiv 1$ in (2.5). Then

$$\mu(S) \le Cw(S) \tag{2.6}$$

Let $a > \operatorname{ess.inf}_{B(x,r)} w(x)$ and put

$$S_a = \{y \in B(x,r) : w(y) < a\}.$$

Then $\mu(S_a) > 0$. Thus by (2.6), $\mu(S_a) \le Cw(S_a)$ and so

$$1 \le \frac{C}{\mu(S_a)} \int_{S_a} w(x)d\mu(x) \le Ca.$$

Thus $C \le w(y)$ for a.e. $y \in B(x,r)$, which gives (2.1).

Next, we deal with the case $p > 1$. Take $f(x) = f(x)^p w(x)$ so that $f(x) = w(x)^{-\frac{1}{p-1}}$. Then by (2.3) we obtain

$$\left(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(y)^{-\frac{1}{p-1}} d\mu(y) \right)^p \le \frac{C}{\nu(B(x,r))} \int_{B(x,r)} w(y)^{-\frac{1}{p-1}} d\mu(y).$$

Hence

$$\left(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(y)^{-\frac{1}{p-1}} d\mu(y) \right)^{p-1} \le C.$$

Conversely, let $p = 1$ and assume that (2.1). Then, we get

$$\begin{aligned} \nu(\{M_\nu f > \lambda\} &\le \frac{C}{\lambda} \int_{\mathbb{R}^n} f(x)dx \\ &\le \frac{C}{\lambda} \int_{\mathbb{R}^n} f(x)w(x)dx, \end{aligned}$$

which shows (1.3). Next, let $p \ge 1$ and assume that (2.2). By Hölder's inequality, we have

$$\begin{aligned} f_{B(x,r)} &= \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(y)w(y)^{1/p} w(y)^{-1/p} d\mu(y) \\ &\le \left(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(y)^p w(y) d\mu(y) \right)^{1/p} \\ &\times \left(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(y)^{-\frac{1}{p-1}} d\mu(y) \right)^{(p-1)/p} \end{aligned}$$

Therefore, by (2.2) we get

$$\begin{aligned}
&f^p_{B(x,r)}\nu(B(x,r)) \\
&\le \left(\int_{B(x,r)} f(y)^p w(y)d\mu(y)\right)\left(\frac{1}{\nu(B(x,r))}\int_{B(x,r)} w(x)^{-\frac{1}{p-1}}d\mu(y)\right)^{p-1} \\
&\le C\int_{B(x,r)} f(y)^p w(y)d\mu(y),
\end{aligned}$$

which is (2.4). Hence (2.2) implies (2.4). Now assume that $f \in L^p(wd\mu)$. Here we can assume that $f \geq 0$ without loss of generality. By (2.4), we have $L^p_{loc}(wd\mu) \subset L^1_{loc}(d\mu)$. Now we can also assume that $f \in L^1(d\mu)$. In fact, we can write $f = \lim_{j\to\infty} f_j$, where $f_j = f\chi_{B(p,r)}$, where p is a some fixed point in X. If (1.2) holds for any f_j, then passing to the limit (1.2) holds for $f \in L^1(X, d\mu)$. So it is sufficient to estimate (1.2) when $f \in L^1(X, d\mu)$. Pick an arbitrary but fixed point p and $r > 0$. For each $x \in E_\lambda \cap B(x_o, r)$, there exists a ball $B(x, r(x))$ such that

$$\frac{1}{\nu(B(x,r(x)))}\int_{B(x,r(x))} f(y)d\mu(y) > \lambda. \tag{2.7}$$

Then there exists a countable subfamily of balls $\{B(x_j, r_j)\}$ satisfying the covering lemma. Hence by (2.3), the doubling property of ν, and (2.7), it follows that

$$\begin{aligned}
\nu(E_\lambda \cap B(x_o,r)) &\le C\sum_{j=1}^{\infty}\nu(B(x_j,4Kr_j)) \\
&\le C\sum_{j=1}^{\infty}\nu(B(x_j,r_j)) \qquad (2.8) \\
&\le C\sum_{j=1}^{\infty}\left(\frac{1}{\nu(B(x_j,r_j))}\int_{(B(x_j,r_j)} f(y)d\mu(y)\right)^{-p} \\
&\qquad \left(\int_{(B_j,r_j)} f(y)^p w(y)d\mu(y)\right)
\end{aligned}$$

$$\leq \frac{C}{\lambda^p} \sum_{j=1}^{\infty} \int_{B(x_j r_j)} f(y)^p w(y) d\mu(y)$$
$$\leq \frac{C}{\lambda^p} \int_X f(y)^p w(y) d\mu(y).$$

Since the constant C in (2.8) is independent of x_0 and r, by letting $r \uparrow \infty$, we obtain (1.3). This completes the proof. □

COROLLARY. *Let $0 < \eta < 1$ and $X = \mathbb{R}^n$, $d(x,y) = |x-y|$, and $d\mu = dx$ the n-dimensional Lebesgue measure. Then*

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\eta}} \int_{B(x,r)} |f(y)| dy,$$

where $B(x,r)$ is a Euclidean ball centered at x and with diameter $r > 0$. In this case, (1.3) holds if and only if (2.1) or (2.2) holds for each corresponding p.

Proof. Put $\nu(B(0,r)) = |B(0,r)|^{1-\eta}$, where $0 < \eta < 1$. Since

$$|B(0,2r)|^{1-\eta} = C(2r)^{1-\eta} = C2^{1-\eta} r^{1-\eta} \leq 2Cr^{1-\eta},$$

ν is a doubling measure. □

References

[1] A. S. Besicovitch, *A general form of the covering principle and relative differentiation of additive functions*, Proc. Cambridge Philos. Soc. **41** (1945), 103-110.

[2] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569-645.

[3] M. de Guzmán, *Real-Variable Methods in Fourier Analysis*, vol. 46, Math. Study, North-Holland, 1981.

[4] José Garcia-Cuerva and José L. Lubio de Francia, *Weighted Normed Inequalities and Related Topics*, North-Holland, 1985.

[5] A. P. Morse, *Perfect blankets*, Trans. Amer. Math. Soc. **6** (1947), 418-442.

[6] P. Sjögren, *Admissible convergence of Poisson integrals in symmetric spaces*, Ann. of Math. **124** (1986), 313-335.

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