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INEQUALITIES FOR THE AREA OF CONSTANT RELATIVE BREADTH CURVES

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ABSTRACT. We obtain an efficient upper bound of the area of convex curves of constant relative breadth in the Minkowski plane. The estimation is given in terms of the Minkowski arc length of pedal curve of original curve.

1. Introduction

Bodies of constant breadth are of general interest ([1], [2], [4], [5], [6], [7], [9], [10], [11], [13]). Many mathematicians have studied geometric inequalities for convex bodies ([3], [4], [5], [8]). Chakerian and Groemer ([10]), Gruber and Wills ([12]), and Santaló ([15]) are good references in this line.

In this paper, we use the concept of a pedal curve and obtain a Minkowski arc length element for non-convex closed plane curve in the Minkowski plane to calculate an upper bound of the area of constant relative breadth curves and obtain an inequality for the area of a curve C of constant relative breadth in the Minkowski plane:

$$A(C) \leq 2A(P) - \frac{\pi L^2(P)}{A_{2,2}^2 \left(\frac{m^2 m_{\alpha'}^2}{M^2 M_{\alpha'}^2}\right) L_e^2(I)},$$

where P is a pedal curve of C and A is the Euclidean area and L is the Minkowski arc length and L_e is the Euclidean arc length and m and M are

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minimal and maximal values of the radial function of the isoperimetrix I and α is the angle function of contact points to C.

The equality holds if and only if C is homothetic to a pedal curve of the isoperimetrix.

2. Preliminaries

Let C be a closed plane curve with O in its interior. If O bisects each chord of C through O, then we say that O is a center of C. Let U be a centrally symmetric closed convex curve with center at the origin O of the Euclidean plane R^2. We shall assume throughout that U is smooth and has positive finite curvature everywhere. Then a usual metric d on R^2 defines a Minkowski metric m using the formula $m(x,y) = \frac{d(x,y)}{r(x,y)}$, where $d(x,y)$ is the Euclidean distance from x to y, and $r(x,y)$ is the radius of U in the direction of a vector $x-y$. The set of points of R^2 together with metric m is the Minkowski plane, denoted by M^2. Certainly U is the unit circle in M^2 and it shall be referred to as the indicatrix. Let $r(\theta) = \rho^{-1}(\theta)$, where $r(\theta)$ is the radial function of U in the direction θ. Then the function $\rho(\theta)$ is the support function of a closed convex curve I called the isoperimetrix. In fact, the curve I is the polar reciprocal of U, with respect to the Euclidean unit circle, rotated through deg 90 ([3]).

Now we have the following definition for the Minkowski breadth.

DEFINITION 1. Let C be a closed convex curve in the Minkowski plane M^2. Then the relative breadth $\omega(\theta)$ in the direction θ is the distance between two parallel lines of support to C which are perpendicular to the direction θ and which contain C between them. C is of constant relative breadth if its relative breadth is independent of the direction.

The relative breadth of a closed convex curve C in a given direction θ is $\frac{h(\theta)+h(\theta+\pi)}{p(\theta)}$, where $h(\theta)$ and $p(\theta)$ are the support functions of C and U, respectively. Thus all the homotheties ωU of U are curves of constant relative breadth 2ω. The authors consider an example of non-trivial curves of constant relative breadth in [14].

3. The dual Minkowski plane M^{2^*}

Following Chakerian [8], we parametrize U by twice its sectorial area, θ, and write the equation of U as

$$t = t(\theta), -\pi \leq \theta \leq \pi, m(t) = m(t, O) = 1. \tag{1}$$

Then the trace of $n(\theta)$ defined by $n(\theta) = \frac{dt(\theta)}{d\theta}, -\pi \leq \dot{\theta} \leq \pi$, is the isoperimetrix I. Let G be a line parallel to the direction $t(\theta)$ in (1). Then the equation of G can be given by the formula:

$$|[t(\theta), X]| = P, \tag{2}$$

where $[X, Y] = x_1y_2 - x_2y_1$ for $X = (x_1, x_2), Y = (y_1, y_2)$ and $|\cdot|$ denotes the absolute value of the number. We shall denote the line G by $G = G(P, \theta)$. In fact, $[X, JY]$ is equivalent to $\langle X, Y\rangle$ where J is rotation by $\frac{\pi}{2}$ and $\langle , \rangle$ is the Euclidean dot product. The authors investigate geometry of the number P in [4].

THEOREM 1. *Let $G(P, \theta)$ be a line given by the equation* (2). *Then the point $Pn(\theta)$ lies on the line $G(P, \theta)$ and P is the Minkowski distance in the new Minkowski plane with the indicatrix I from the origin to the line $G(P, \theta)$.*

Proof. See [4]. □

In Theorem 1 we define the new Minkowski plane with the isoperimetrix I as its indicatrix. We call this new plane *the dual plane* of the original Minkowski plane and denote it by M^{2^*}. Also we call the distance m^* on M^{2^*} the *dual distance.* We shall denote the relative breadth in M^{2^*} by $Br^*(C, \theta)$.

Now we consider some properties of curves of constant relative breadth in the new Minkowski plane M^{2^*}.

THEOREM 2. *Let C be a plane curve of constant relative breadth $Br^*(C, \theta) = \omega$ in M^{2^*}. Then*

$$A(C) + A(C, -C) = \frac{\omega^2}{2}T,$$

where $-C$ is a curve obtained by rotating C through deg 180 and $A(,)$ denotes the mixed area.

Proof. See [14]. □

4. A constant relative breadth curve

Let C be a differentiable closed convex curve with the origin in its interior. For the direction $\theta, -\pi \leq \theta \leq \pi$, let $h(\theta)$ be the support function of C. That is, $h(\theta)$ is the distance of the origin from the tangent of C at the point $q(\theta)$ where the exterior normal of C has direction θ. It is well known that the Euclidean line element of C at $q(\theta)$ is $(h(\theta)+h''(\theta))d\theta$. Assume $\theta_1 < \theta_2$. Then the line through $q(\theta_1)$ and perpendicular to the direction θ_2 must meet the different point of C from $q(\theta_1)$. Thus $q(\theta_1) \neq q(\theta_2)$.

Conversely, assume that $\theta_1 = \theta_2$ for different two points $q(\theta_1), q(\theta_2)$ on C. Then the line segment $\overline{q(\theta_1)q(\theta_2)}$ is contained in C. This is a contradiction to the fact that C is differentiable. Consequently, we have that the function $q : \theta \mapsto q(\theta), \theta \in [-\pi, \pi]$ is a one-to-one correspondence. Thus the Minkowski length $L(C)$ of C is

$$L(C) = \int_{-\pi}^{\pi} (h(\theta) + h''(\theta))\rho(\theta \pm \frac{\pi}{2})d\theta. \tag{3}$$

If C is not convex, then we cannot use the equation (3) for the Minkowski length of a closed curve because the function q above is not one-to-one.

The notion of a pedal curve was introduced by C. Maclaurin.

DEFINITION 2. For a simple closed plane convex curve C with the origin O in its interior, the curve P whose radial function is equal to the support function of C is called a pedal curve of C.

L. Xiao-hua showed the following lemma.

LEMMA 1. *If f and g are positive real valued functions and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 0$ and $0 < m \leq \frac{f}{g} \leq M$ hold, then*

$$\left(\int f\right)^{\frac{1}{p}} \left(\int g\right)^{\frac{1}{q}} \leq A_{p,q}\left(\frac{m}{M}\right) \int f^{\frac{1}{p}} g^{\frac{1}{q}},$$

where

$$A_{p,q}(t) = \frac{1}{p^{\frac{1}{p}} q^{\frac{1}{q}}} \frac{(1-t)t^{\frac{1}{pq}}}{(1-t^{\frac{1}{p}})^{\frac{1}{p}}(1-t^{\frac{1}{q}})^{\frac{1}{q}}}.$$

Proof. See [16] □

With notations $\tilde{I}$ for a pedal curve of the isoperimetrix I and $\tilde{T}$ for the area enclosed by $\tilde{I}$, we have our main result in the following:

THEOREM 3. *If C is a differentiable curve of constant relative breadth ω in M^{2^*} and P is a pedal curve of C, then*

$$A(C) \leq 2A(P) - \frac{\pi L^2(P)}{A_{2,2}^2\left(\frac{m^2 m_{\alpha'}^2}{M^2 M_{\alpha'}^2}\right) L_e^2(I)}. \tag{4}$$

The equality holds if and only if C is homothetic to $\tilde{I}$.

COROLLARY 1. *Let C be a curve of constant breadth ω in the Euclidean plane and P be a pedal curve of C, then*

$$A(C) \leq 2A(P) - \frac{L_e^2(P)}{4\pi A_{2,2}^2\left(\frac{m_{\alpha'}^2}{M_{\alpha'}^2}\right)}, \tag{5}$$

where L_e is the Euclidean arc length. The equality holds if and only if C is a circle.

Proof. In the Euclidean plane, $L_e(I) = 2\pi$ and $\rho = 1$. Thus we have the inequality (5) immediately from the theorem above.

Secondly, in the case C is a circle we have $P = C$ and $m_{\alpha'} = M_{\alpha'} = 1$. Since $A_{2,2}(t) = \frac{1}{2}(1+\sqrt{t})t^{\frac{1}{4}}$, we get $A_{2,2}(1) = 1$. Thus $A(C) = 2A(P) - \frac{L_e^2(P)}{4\pi A_{2,2}^2\left(\frac{m_{\alpha'}^2}{M_{\alpha'}^2}\right)}$. This completes the proof. □

5. Proof of Theorem 3

Note that pedal curve P is not convex in general. Thus we use $\sqrt{x^2(\theta) + x'^2(\theta)}d\theta$ as the Euclidean arclength element of P at $x(\theta)$, where $x(\theta)$ is the radial function of P. Then $\sqrt{x^2(\theta) + x'^2(\theta)} = \sqrt{h^2(\theta) + h'^2(\theta)}$ is the Euclidean length of $\overline{Oq(\theta)}$. Let $\alpha(\theta)$ be the angle of $\overline{Oq(\theta)}$ from the line $\theta = 0$ and extend the function α to the function $\alpha : (-\infty, \infty) \to (-\infty, \infty)$ by the equation $\alpha(\theta + 2k\pi) = \alpha(\theta) + 2k\pi$, where k is an integer. Since C is differentiable the function $\alpha : \theta \mapsto \alpha(\theta)$ from $(-\infty, \infty)$ to

$(-\infty, \infty)$ is an increasing one-to-one correspondence. Thus the Minkowski arc length element of P at $x(\theta)$ is $\sqrt{x^2(\theta) + x'^2(\theta)}\rho \circ \alpha(\theta) d\theta$. Thus

$$\begin{aligned} L(P) &= \int_{-\pi}^{\pi} \sqrt{x^2(\theta) + x'^2(\theta)}\rho \circ \alpha(\theta) d\theta \\ &= \int_{-\pi}^{\pi} \sqrt{h^2(\theta) + h'^2(\theta)}\rho \circ \alpha(\theta) d\theta \end{aligned}$$

Now by using Buniakowsky-Schwarz inequality, $\left(\int fg\right)^2 \leq (\int f^2)(\int g^2)$, we have

$$\begin{aligned} &\int_{-\pi}^{\pi} \sqrt{h^2(\theta) + h'^2(\theta)}\rho \circ \alpha(\theta) d\theta \\ &\leq \sqrt{\int_{-\pi}^{\pi} \{h^2(\theta) + h'^2(\theta)\} d\theta \int_{-\pi}^{\pi} (\rho \circ \alpha)^2(\theta) d\theta} \end{aligned} \tag{6}$$

Thus

$$\begin{aligned} &L(P) \\ &\leq \sqrt{B \int_{-\pi}^{\pi} \{h^2(\theta) + h'^2(\theta)\} d\theta} \\ &= \sqrt{B}\sqrt{\omega^2 \int_0^{\pi} \{\rho^2(\theta) + \rho'^2(\theta)\} d\theta + 2A(C, -C) - 4\int_0^{\pi} h(\theta)h(\theta + \pi) d\theta}, \end{aligned}$$

where $B = \int_{-\pi}^{\pi} (\rho \circ \alpha)^2(\theta) d\theta$. Because C is of constant relative breadth ω in M^{2^*}, we have

$$\begin{aligned} \int_0^{\pi} h(\theta)h(\theta + \pi) d\theta &= \frac{\omega^2}{2}\int_0^{\pi} \rho^2(\theta) d\theta - \frac{1}{2}\int_{-\pi}^{\pi} h^2(\theta) d\theta \\ &= \frac{\omega^2}{2}\tilde{T} - A(P). \end{aligned}$$

Thus by Theorem 2

$$L(P) \leq \sqrt{2B(2A(P) - A(C))}. \tag{7}$$

Let M and m be maximal and minimal radius of I, respectively. Then trivially $m \leq \rho \circ \alpha(\theta) \leq M$ for all $\theta \in [-\pi, \pi]$. We can compute that

$q(\theta) = (h(\theta)\cos\theta - h'(\theta)\sin\theta, h(\theta)\sin\theta + h'(\theta)\cos\theta)$. Thus the equation of the line through O and $q(\theta)$ is

(8) $\quad (h(\theta)\sin\theta + h'(\theta)\cos\theta)x + (h'(\theta)\sin\theta - h(\theta)\cos\theta)y = 0.$

Thus $(h(\theta)\cos\theta - h'(\theta)\sin\theta)tan(\alpha(\theta)) = h(\theta)\sin\theta + h'(\theta)\cos\theta$ for $\alpha(\theta) \neq \frac{\pi}{2} + k\pi$, where k is an integer. Thus

$$\alpha'(\theta) = \frac{h(\theta)(h(\theta) + h''(\theta))}{h^2(\theta) + h'^2(\theta)}$$

for $\alpha(\theta) \neq \frac{\pi}{2} + k\pi$. Since C has no line segment, $h + h''$ is continuous. Extend α' to the domain where $\alpha(\theta) = \frac{\pi}{2} + k\pi$ by the equation

$$\alpha'\left(\alpha^{-1}\left(\frac{\pi}{2} + k\pi\right)\right) = \frac{h(h+h'')}{h^2 + h'^2}\left(\alpha^{-1}\left(\frac{\pi}{2} + k\pi\right)\right).$$

Then α' is continuous on R. Thus α' has its maximal value $M_{\alpha'}$ and minimal value $m_{\alpha'}$ on $[-\pi, \pi]$. Thus

$$0 < \left(\frac{m}{M_{\alpha'}}\right)^2 \leq \frac{(\rho \circ \alpha)^2(\theta)}{(\alpha')^2(\theta)} \leq \left(\frac{M}{m_{\alpha'}}\right)^2.$$

By Lemma 1 we have

$$\begin{aligned} &\sqrt{\int_{-\pi}^{\pi} (\rho \circ \alpha)^2(\theta) d\theta \int_{-\pi}^{\pi} (\alpha')^2(\theta) d\theta} \\ &\leq A_{2,2}\left(\frac{m^2 m_{\alpha'}^2}{M^2 M_{\alpha'}^2}\right) \int_{-\pi}^{\pi} (\rho \circ \alpha)(\theta)\alpha'(\theta) d\theta \\ &= A_{2,2}\left(\frac{m^2 m_{\alpha'}^2}{M^2 M_{\alpha'}^2}\right) L_e(I). \end{aligned}$$

Using Schwarz Inequality we have $\sqrt{\int_{-\pi}^{\pi} (\alpha')^2(\theta) d\theta} \geq \sqrt{2\pi}$. Thus

$$\sqrt{\int_{-\pi}^{\pi} (\rho \circ \alpha)^2(\theta) d\theta} \leq \frac{A_{2,2}\left(\frac{m^2 m_{\alpha'}^2}{M^2 M_{\alpha'}^2}\right) L_e(I)}{\sqrt{2\pi}}.$$

Consequently we get

$$L(P) \leq \sqrt{\frac{A_{2,2}^2\left(\frac{m^2 m_{\alpha'}^2}{M^2 M_{\alpha'}^2}\right) L_e^2(I)(2A(P) - A(C))}{\pi}}.$$

This yields the inequality (4).

Now we prove the second part of our theorem. The equality holds in (6) if and only if $\sqrt{h^2(\theta)+h'^2(\theta)}$ is proportional to $\rho\circ\alpha(\theta)$. But the trace of $\rho\circ\alpha(\theta)$ is a pedal curve $\tilde{I}$ of the isoperimetrix I. We may assume that a support line l to C at a point $q(\theta)$ on C is perpendicular to the direction θ at a point q' on l. Then $d(O,q')=h(\theta)$ and $d(q(\theta),q')=|h'(\theta)|$. Thus $d(O,q(\theta))=\sqrt{h^2(\theta)+h'^2(\theta)}$. Since C is a differentiable curve, $q(\theta)$ has a unique θ modulo 2π associated with it and θ makes a complete circuit. If we set $y(\theta)=d(O,q(\theta))=\sqrt{h^2(\theta)+h'^2(\theta)}$, then $y(\theta)$ is also an equation of C. This completes the proof. □

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