Bull. Korean Math. Soc. 36 (1999), No. 1, pp. 1–13

# CONTROLLABILITY OF SECOND ORDER SEMILINEAR VOLTERRA INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

K. BALACHANDRAN, J. Y. PARK AND S. MARSHAL ANTHONI

ABSTRACT. Sufficient conditions for controllability of semilinear second order Volterra integrodifferential systems in Banach spaces are established using the theory of strongly continuous cosine families. The results are obtained by using the Schauder fixed point theorem. An example is provided to illustrate the theory.

## 1. Introduction

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the concept to infinite dimensional systems in Banach Spaces with bounded operators. Lasiecka and Triggiani [4] have studied the exact controllability of abstract semilinear equations. Naito [7,8] has studied the controllability for semilinear systems and nonlinear Volterra integrodifferential systems. Quinn and Carmichael [10] have shown that the controllability problem in Banach spaces can be converted into one of a fixed-point problem for a single-valued mapping. Balachandran et al [1] established sufficient conditions for controllability of nonlinear integrodifferential systems in Banach spaces.

In many cases it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. A useful tool for the study of abstract second order equations

Received December 3, 1997. Revised June 1, 1998.
1991 Mathematics Subject Classification: 93B05.
Key words and phrases: controllability, semilinear Volterra integrodifferential system, fixed point theorem.

is the theory of strongly continuous cosine families. We will make use of some of the basic ideas from cosine family theory and the theory of fractional powers [11,12,2]. Motivation for second order systems can be found in [3,5,6]. Because of its importance recently Park et al.[9] have discussed the controllability of second order nonlinear systems in Banach spaces with the help of Schauder's fixed point theorem. The purpose of this paper is to study the controllability of semilinear second order Volterra integrodifferential systems in Banach spaces by using the Schauder fixed point theorem.

## 2. Preliminaries

We consider the abstract semilinear second order control system

$$\begin{aligned} x''(t) &= Ax(t) + \int_0^t g(t,s,x(s))ds + Bu(t), \qquad t \in J = [0,T], \\ (1) \quad x(0) &= x_0, \qquad x'(0) = y_0, \end{aligned}$$

where the state $x(.)$ takes values in the Banach space $X$, $x_0, y_0 \in X$, $A$ is the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators in $X$, $g$ is a nonlinear unbounded mapping from $J \times J \times X$ to $X$, $B$ is a bounded linear operator from $U$ to $X$ and the control function $u(.)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with $U$ as a Banach space.

DEFINITION 1. [13] A one parameter family $C(t)$, $t \in R$, of bounded linear operators in the Banach space $X$ is called a strongly continuous cosine family iff

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in R$;
(ii) $C(0) = I$;
(iii) $C(t)x$ is continuous in $t$ on $R$ for each fixed $x \in X$.

We define the associated sine family $S(t)$, $t \in R$, by

$$S(t)x = \int_0^t C(s)x ds, \qquad x \in X, \quad t \in R.$$

We make the following assumption on $A$.

$(H_1)$ $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from $X$ into itself.

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x \Big|_{t=0}, \qquad x \in D(A),$$

where
$D(A) = \{ x \in X\colon C(t)x \text{ is twice continuously differentiable in } t \}$ .

We define
$E = \{ x \in X\colon C(t)x \text{ is once continuously differentiable in } t \}$ .

LEMMA 2. [13] *Let $(H_1)$ hold. Then*
(i) *there exist constants $M \geq 1$ and $\omega \geq 0$ such that*
$\|C(t)\| \leq Me^{\omega|t|}$ *and* $\|S(t)\| \leq Me^{\omega|t|}$ *for* $t \in R$;
(ii) $S(t)X \subset E$ *and* $S(t)E \subset D(A)$ *for* $t \in R$;
(iii) $\frac{d}{dt} C(t)x = AS(t)x$ *for* $x \in E$ *and* $t \in R$;
(iv) $\frac{d^2}{dt^2} C(t)x = AC(t)x$ *for* $x \in D(A)$ *and* $t \in R$.

It is proved in [2] that for $0 \leq \alpha \leq 1$ the fractional powers $(-A)^\alpha$ exist as closed linear operators in $X$, $D((-A)^\alpha) \subset D((-A)^\beta)$ for $0 \leq \beta \leq \alpha \leq 1$, and $(-A)^\alpha(-A)^\beta = (-A)^{\alpha+\beta}$ for $0 \leq \alpha + \beta \leq 1$.

We assume in addition

$(H_2)$ for $0 \leq \alpha \leq 1$, $(-A)^\alpha$ maps onto $X$ and is 1-1, so that $D((-A)^\alpha)$ is a Banach space when endowed with the norm $\|x\|_\alpha = \|(-A)^\alpha x\|$, $x \in D((-A)^\alpha)$. We denote this Banach space by $X_\alpha$. We further assume that $A^{-1}$ is compact.

We require the following lemmas.

LEMMA 3. [13] *Suppose* $(H_1)$ *hold. Then the following are true.*
(i) *For* $0 < \alpha < 1$, $(-A)^{-\alpha}$ *is compact iff* $A^{-1}$ *is compact,*
(ii) *for* $0 < \alpha < 1$ *and* $t \in R$, $(-A)^{-\alpha}C(t) = C(t)(-A)^{-\alpha}$
*and* $(-A^{-\alpha})S(t) = S(t)(-A)^{-\alpha}$.

LEMMA 4. [13] *Let* $(H_1)$ *hold, let* $v : R \to X$ *such that* $v$ *is continuously differentiable and let* $q(t) = \int_0^t S(t-s)v(s)ds$. *Then*

(i) $q$ *is twice continuously differentiable and for* $t \in R$,

$$q(t) \in D(A), \quad q'(t) = \int_0^t C(t-s)v(s)ds,$$

*and*

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t);$$

(ii) *for* $0 \le \alpha \le 1$ *and* $t \in R$, $(-A)^{\alpha-1}q'(t) \in E$.

We make the following assumptions:

$(H_3)$ $g : J \times J \times D(A) \to X$ is continuous where $D(A)$ is an open subspace of $X_\alpha$, for some $\alpha \in$[0,1);

$(H_4)$ $g_1 : J \times J \times D(A) \to X$ is continuous where $g_1$ denotes the derivative of $g$ with respect to its first variable;

$(H_5)$ $Bu$ is continuously differentiable and Bu(0)=0;

$(H_6)$ The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

has a bounded invertible operator $W^{-1} : X \to L^2(J, U)/kerW$.

For the system (1), a mild solution can be written as (see [13])

$$
\begin{aligned}
x(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s) \int_0^s g(s,\tau,x(\tau))d\tau ds \\
&\quad + \int_0^t S(t-s)Bu(s)ds. \qquad (2)
\end{aligned}
$$

DEFINITION 5. The system (1) is said to be controllable on $J = [0, T]$ if for every $x_0, x_1 \in D(A)$ and $y_0 \in E$ there exists a control $u \in L^2(J, U)$ such that the solution $x(.)$ of (1) satisfies $x(T) = x_1$.

## 3. Main result

THEOREM. Suppose $(H_1)$-$(H_6)$ hold. Then the system (1) is controllable on $J$.

*Proof.* Using the assumption $(H_6)$, for an arbitrary function $x(.)$ we define the control

$$
u(t) = W^{-1}[x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-s) \int_0^s g(s,\tau,x(\tau))d\tau ds](t).
$$

Using this control we shall now show that the operator defined by

$$
\begin{aligned}
(Gx)(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s) \int_0^s g(s,\tau,x(\tau))d\tau ds \\
&\quad + \int_0^t S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\
&\quad - \int_0^T S(T-\theta) \int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta](s)ds, \quad t \in J
\end{aligned}
$$

has a fixed point. This fixed point is then a solution of the equation (2). □

Clearly, $(Gx)(T) = x_1$, which means that the control $u$ steers the system from the initial state $x_0$ to $x_1$ in time $T$, provided we obtain a fixed point of the nonlinear operator $G$.

For $\gamma > 0$, let $\quad N_\gamma(x_0) = \{x \in X_\alpha : \|x_0 - x\| < \gamma\}.$

Let $\phi(t) = C(t)x_0 + S(t)y_0$. Then $\phi : R \to X_\alpha$ is continuous. Now choose $\gamma > 0$ and $T > 0$ such that

$$N_\gamma(x_0) \subset D(A); \tag{3}$$

for $s, \tau \in [0, T]$ and $x \in N_\gamma(x_0)$,

$$\|g(s, \tau, x)\| \leq 1 \quad \textit{and} \quad \|g_1(s, \tau, x)\| \leq 1; \tag{4}$$

for $t \in [0, T]$,

$$\|\phi(t) - x_0\|_\alpha < \gamma/2; \tag{5}$$

for $t \in [0, T]$ and $x_2, x_3, x_4, x_5 \in N_\gamma(x_0)$,

$$\begin{aligned} &\Bigg\| (-A)^{\alpha-1} \Bigg\{ \int_0^t g(t, s, x_2) ds - \int_0^t C(t-s) \, (g(s, s, x_3) \\ &\quad + \int_0^s g_1(s, \tau, x_4) d\tau) ds + BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta) \int_0^\theta g(\theta, \tau, x_5) d\tau d\theta](t) \\ &\quad - \int_0^t C(t-s) (\frac{d}{ds} BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta) \int_0^\theta g(\theta, \tau, x_5) d\tau d\theta](s)) ds \Bigg\} \Bigg\| < \gamma/2. \end{aligned} \tag{6}$$

Let $K$ be a closed bounded convex subset of $Z = C([0, T] : X_\alpha)$ with norm $\|.\|_Z$, defined by

$$K = \{x \in Z : \|x - \phi\| \le \gamma/2\}.$$

Notice that for $x \in K$ and $t \in [0,T]$, $\quad x(t) \in D(A)$,

since $\|x(t) - x_0\|_\alpha \le \|x - \phi\|_Z + \|\phi(t) - x_0\|_\alpha \le \gamma/2 + \gamma/2$.

Define the transformation $G$ on $K$ by

$$\begin{aligned}(Gx)(t) &= \phi(t) + \int_0^t S(t-s)\int_0^s g(s,\tau,x(\tau))d\tau ds \\ &\quad + \int_0^t S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta)\int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta](s)ds, \quad t \in J.\end{aligned}$$

Using $(H_2)$ and (6) we see that for $t \in [0,T]$,

$$\begin{aligned}&\|(Gx)(t) - \phi(t)\|_\alpha \\ &= \Bigg\|(-A)^{\alpha-1}\Bigg\{\int_0^t g(t,s,x(s))ds \\ &- \int_0^t C(t-s)(g(s,s,x(s)) + \int_0^s g_1(s,\tau,x(\tau))d\tau)ds \\ &+ BW^{-1}\left[x_1 - \phi(T) - \int_0^T S(T-\theta)\int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta\right](t) \\ &- \int_0^t C(t-s)\Bigg(\frac{d}{ds}BW^{-1}[x_1 - \phi(T) \\ &\qquad - \int_0^T S(T-\theta)\int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta](s)\Bigg)ds\Bigg\}\Bigg\| < \gamma/2.\end{aligned}$$

Further $Gx$ is continuous as a function from $[0,T]$ to $X_\alpha$. Thus $G$ maps $K$ into $K$.
We next show that $G$ is continuous. By $(H_3)$ and $(H_4)$, given $\epsilon > 0$ there exists a $\delta > 0$ such that for $x_1, x_2 \in K$, $\|x_1 - x_2\|_Z < \delta$ and $s \in [0,T]$, we have

$$\sup_{0 \le \tau \le T} \|g(s,\tau,x_1(\tau)) - g(s,\tau,x_2(\tau))\| < \epsilon$$

$$\sup_{0 \le \tau \le T} \|g_1(s,\tau,x_1(\tau)) - g_1(s,\tau,x_2(\tau))\| < \epsilon$$

Thus for $x_1, x_2 \in K$ and $t \in [0,T]$,

$$\begin{aligned}
&\|(Gx_1)(t) - (Gx_2)(t)\|_\alpha \\
&= \Bigg\|(-A)^{\alpha-1}\Bigg\{\int_0^t (g(t,s,x_1(s)) - g(t,s,x_2(s)))ds \\
&- \int_0^t C(t-s)(g(s,s,x_1(s)) - g(s,s,x_2(s)))ds \\
&- \int_0^t C(t-s)\int_0^s (g_1(s,\tau,x_1(\tau)) - g_1(s,\tau,x_2(\tau)))d\tau ds \\
&+ BW^{-1}\left[-\int_0^T S(T-\theta)\int_0^\theta (g(\theta,\tau,x_1(\tau)) - g(\theta,\tau,x_2(\tau)))d\tau d\theta\right](t) \\
&- \int_0^t C(t-s)\left(\frac{d}{ds}BW^{-1}\left[-\int_0^T S(T-\theta)\right.\right. \\
&\qquad \left.\left.\int_0^\theta (g(\theta,\tau,x_1(\tau)) - g(\theta,\tau,x_2(\tau)))d\tau d\theta\right](s)\right)ds\Bigg\}\Bigg\| < N\epsilon
\end{aligned}$$

for some constant $N > 0$ and the continuity of G follows immediately.
We next show that the set $\{Gx : x \in K\}$ is equicontinuous as a collection of functions in $Z$.

For $x \in K$ and $0 \leq t \leq t^* \leq T$,

$$
\begin{aligned}
&\|(Gx)(t) - (Gx)(t^*)\|_\alpha \\
&\leq \|(C(t) - C(t^*))(-A)^\alpha x_0\| + \|A(S(t) - S(t^*))(-A)^{\alpha-1}y_0\| \\
&\quad + \left\| \int_0^t (C(t-s) - C(t^*-s))(-A)^{\alpha-1}(g(s,s,x(s)) \right. \\
&\qquad\qquad \left. + \int_0^s g_1(s,\tau,x(\tau))d\tau)ds \right\| \\
&\quad + \|(-A)^{\alpha-1}\| \left\| \int_t^{t^*} (C(t^*-s)(g(s,s,x(s)) \right. \\
&\qquad\qquad \left. + \int_0^s g_1(s,\tau,x(\tau))d\tau)ds \right\| \\
&\quad + \|(-A)^{\alpha-1}\| \left\{ \left\| \int_0^t \int_t^{t^*} g_1(\tau,s,x(s))d\tau ds \right\| + \left\| \int_t^{t^*} g(t^*,s,x(s))ds \right\| \right\} \\
&\quad + \left\| (-A)^{\alpha-1} \left\{ BW^{-1} \left[ x_1 - \phi(T) \right. \right. \right. \\
&\qquad\qquad \left. - \int_0^T S(T-\theta) \int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta \right](t) \\
&\quad \left. \left. - BW^{-1} \left[ x_1 - \phi(T) - \int_0^T S(T-\theta) \int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta \right](t^*) \right\} \right\| \\
&\quad + \|(-A)^{\alpha-1}\| \left\| \int_0^t (C(t-s) - C(t^*-s)) \Big( \frac{d}{ds} BW^{-1} \left[ x_1 - \phi(T) \right. \right. \\
&\qquad\qquad \left. \left. - \int_0^T S(T-\theta) \int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta \right](s) \Big) ds \right\| \\
&\quad + \|(-A)^{\alpha-1}\| \left\| \int_t^{t^*} (C(t^*-s) \frac{d}{ds} BW^{-1} \left[ x_1 - \phi(T) \right. \right. \\
&\quad \left. \left. - \int_0^T S(T-\theta) \int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta \right](s) ds \right\| \to 0.
\end{aligned}
$$

as $|t - t^*| \to 0$ uniformly for $x \in K$,

by virtue of (4),$(H_5)$, the fact that $(-A)^{\alpha-1}$ is compact from $X$ to $X$ and the fact that $C(t)$ is uniformly continuous in finite $t$-intervals on compact subsets of $X$. Thus the set $\{Gx : x \in K\}$ is equicontinuous.

Finally, we show that for each fixed $t \in [0,T]$, the set $\{(Gx)(t) : x \in K\}$ is precompact in $X_\alpha$. Since $(-A)^{-\beta} : X \to X_\alpha$ is compact where $\alpha < \beta$, it suffices to show that $\{(-A)^{-\beta}(Gx - \phi)(t) : x \in K\}$ is bounded in $X$ for $\alpha < \beta \leq 1$.
By $(H_2)$ we have

$$\begin{aligned}
&\|(-A)^{\beta}(Gx-\phi)(t)\| \\
&\leq \left\|(-A)^{\beta-1}\int_0^t (g(t,s,x(s))ds\right\| \\
&\quad + \left\|(-A)^{\beta-1}\int_0^t C(t-s)(g(s,s,x(s)) + \int_0^s g_1(s,\tau,x(\tau))d\tau)ds\right\| \\
&\quad + \|(-A)^{\beta-1}\|\left\{\left\|BW^{-1}\left[x_1 - \phi(T)\right.\right.\right. \\
&\qquad\qquad \left.\left. - \int_0^T S(T-\theta)\int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta\right](t)\right\| \\
&\quad + \left\|\int_0^t C(t-s)\left(\frac{d}{ds}BW^{-1}\left[x_1-\phi(T)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \int_0^T S(T-\theta)\int_0^\theta g(\theta,\tau,x(\tau))d\tau d\theta\right](s)\right)ds\right\|\right\}
\end{aligned}$$

and the boundedness follows from (4).

By Schauder's fixed point theorem, $G$ has a fixed point in $K$. Any fixed point of $G$ is a mild solution of (1) on $J$ satisfying $(Gx)(t) = x(t) \in D(A)$. Thus the system (1) is controllable on $J$.

## 4. Example

Consider the integro-partial differential system

$$\begin{aligned}
z_{tt}(y,t) &= z_{yy}(y,t) + \int_0^t \sigma(t,s,z(y,s)ds + \mu(y,t), \\
z(0,t) &= z(\pi,t) = 0, \qquad\qquad\qquad\qquad (7) \\
z(y,0) &= z_0(y), \qquad z_t(y,0) = z_1(y), \qquad 0 < y < \pi, \;\; t \in J.
\end{aligned}$$

Let $\sigma : J \times J \times (0,\pi) \to (0,\pi)$ be continuous and also continuously differentiable with respect to its first variable. Let $\mu : (0,\pi) \times J \to (0,\pi)$ be continuous and continuously differentiable with respect to its second variable and $\mu(y,0) = 0$.

Let $X = L^2(0,\pi)$ and let $A : X \to X$ be defined by

$$Aw = w'', \qquad w \in D(A),$$

where $D(A) = \{w \in X : w, w'$ are absolutely continuous, $w'' \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2(w, w_n)w_n, \qquad w \in D(A),$$

where $w_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1, 2, 3, \cdots$ is the orthogonal set of eigenvalues of $A$.

It is easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, in $X$ given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, w_n)w_n, \qquad w \in X,$$

and that the associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(w, w_n)w_n, \qquad w \in X.$$

If we choose $\alpha = \frac{1}{2}$, then $A$ satisfies $(H_2)$, since

$$(-A)^{1/2}w = \sum_{n=1}^{\infty} n(w, w_n)w_n, \qquad w \in D((-A)^{1/2}) = X_{\frac{1}{2}},$$

and

$$(-A)^{-1/2}w = \sum_{n=1}^{\infty} (1/n)(w, w_n)w_n, \qquad w \in X.$$

The compactness of $A^{-1}$ follows from Lemma 3, and the fact that the eigenvalues of $(-A)^{-1/2}$ are $\lambda_n = 1/n, n = 1, 2, \cdots$.

Let $g : J \times J \times X_{\frac{1}{2}} \to X$ be defined by

$$g(t,s,w)(y) = \sigma(t,s,w(y)), \quad w \in X_{\frac{1}{2}}, \quad y \in [0,\pi],$$

and let $Bu : J \supset U \to X$ be defined by

$$(Bu(t))(y) = \mu(y,t), \quad y \in (0,\pi).$$

By Reference [10], we assume that there exists a bounded invertible operator $W^{-1}(.)$ in $L^2(J,U)/kerW$ such that

$$Wu = \int_0^T S(T-s)Bu(s)ds.$$

With this choice of $A$, $g$ and $B$, (1) is the abstract formulation of (7). Further all the conditions of the above theorem are satisfied. Hence the system (7) is controllable on $J$.

REMARK. Further examples with $W : L^2(J,U) \to X$ such that $W^{-1}$ exists and is bounded are discussed in [10].

ACKNOWLEDGEMENT. The work of the first author is supported by CSIR, New Delhi, INDIA. (Grant No: 25(89)97 - EMR-II)

## References

[1] K. Balachandran, J. P. Dauer and P. Balasubramaniam, *Controllability of Nonlinear Integrodifferential Systems in Banach Spaces*, Journal of Optimization Theory and Applications **84** (1995) 83-91.

[2] H. O. Fattorini, *Ordinary Differential Equations in Linear Topological Spaces*, Journal of Differential Equations **5** (1968) 72-105.

[3] W. E. Fitzgibbon, *Global Existence and Boundedness of Solutions to the Extensible Beam Equation*, SIAM Journal of Mathematical Analysis **13** (1982), 739-745.

[4] I. Lasiecka and R. Triggiani, *Exact Controllability of Semilinear Abstract Systems with Application to Waves and Plates Boundary Control Problems*, Applications of Mathematical Optimization **23** (1991) 109-154.

[5] R. C. MacCamy, *A Model for One Dimensional, Nonlinear Viscoelasticity*, Quart. Appl. Math. **35** (1977) 21-33.

[6] ———, *An Integrodifferential Equations with Applications in Heat Flow*, Quart. Appl. Math. **35** (1977) 1-19.

[7] K. Naito, *Controllability of Semilinear Control Systems Dominated by the Linear Part*, SIAM Journal on Control and Optimization **25** (1987) 715-722.

[8] ______, *On Controllability for a Nonlinear Volterra Equation*, Nonlinear Analysis: Theory, Methods and Applications **18** (1992) 99-108.

[9] J. Y. Park and H. K. Han, *Controllability for Some Second Order Differential Equations*, Bulletin of the Korean Mathematical Society **34** (1997) 411-419.

[10] M. D. Quinn and N. Carmichael, *An Approach to Nonlinear Control Problems Using Fixed-Point Methods, Degree Theory, and Pseudo-Inverses*, Numerical Functional Analysis and Optimization **7** (1984-1985) 197-219.

[11] C. C. Travis and G. F. Webb, *Cosine Families and Abstract Nonlinear Second Order Differential Equations*, Acta Mathematica Academiae Scientiarum Hungaricae Tomus **32** (1978) 75-96.

[12] ______, *Compactness, Regularity and Uniform Continuity Properties of Strongly Continuous Cosine Families*, Houston Journal of Mathematics **3** (1977) 555-567.

[13] ______, *An Abstract Second Order Semilinear Second order Volterra Integrodifferential Equation*, SIAM Journal of Mathematical Analysis **10** (1979) 412-424.

K. Balachandran and S. Marshal Anthoni, Department of Mathematics, Bharathiar University, Coimbatore -641 046, INDIA

J. Y. Park, Department of Mathematics, Pusan National University, Pusan 609-735, KOREA