Comm. Korean Math. Soc. **14** (1999), No. 4, pp. 843–850

SOME PRODUCT FORMULAS OF THE GENERALIZED HYPERGEOMETRIC SERIES

YOUNG JOON CHO, TAE YOUNG SEO AND JUNESANG CHOI

ABSTRACT. The object of this paper is to give certain classes of presumably new product formulas involving the generalized hypergeometric series by modifying the elementary method suggested by Bailey.

1. Introduction and Preliminaries

The generalized hypergeometric function with p numerator and q denominator parameters is defined by

$$
{}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] = {}_pF_q\left[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z\right]
= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},
$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial) is defined by, α any complex number,

(1.1)
$$
(\alpha)_n := \begin{cases} \alpha(\alpha+1)\dots(\alpha+n-1), & \text{if } n \in \mathbf{N} := \{1, 2, 3, \dots ,\}, \\ 1 & \text{if } n = 0. \end{cases}
$$

From (1.1), we can easily deduce the following formula:

$$
(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k} \quad (0 \le k \le n). \tag{1.2}
$$

Received February 3, 1999. Revised May 21, 1999.
1991 Mathematics Subject Classification: Primary 33C20; Secondary 33C10, 33C15.
Key words and phrases: hypergeometric series, Dixon's theorem, Kummer's theorem.

The well-known Gamma function Γ has a relationship with the circular function:

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}. \tag{1.3}$$

Gauss's multiplication formula is given as follows:

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} m^{mz-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\left(z + \frac{j-1}{m}\right) \quad (m \in \mathbf{N}). \tag{1.4}$$

The following identity is well known and can readily be deduced from (1.1):

$$(\alpha)_{kn} = k^{nk} \prod_{i=1}^{k} \left(\frac{\alpha+i-1}{k}\right)_n . \tag{1.5}$$

We also recall the following summation formulas for ${}_pF_q$:

$$\begin{gathered} {}_2F_1(a,\, b;\, 1+a-b;\, -1) = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma(1+a)} \\ (\mathrm{Re}(b) < 1\,; 1+a-b \neq 0, -1, -2, \ldots), \end{gathered} \tag{1.6}$$

(see [3, p. 78, Formula 53]);

$$\begin{gathered} {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ 1+a-b,\, 1+a-c \end{matrix};\, 1\right] \\ = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \\ \left(\mathrm{Re}\left(\frac{1}{2}a-b-c\right) > -1\right) \quad \text{(see [2])}. \end{gathered} \tag{1.7}$$

Bailey [1, p. 247] stated the following product formulas associated with the generalized hypergeometric series without proof:

$$e^x\, {}_0F_1\left(\frac{1}{2};\, -\frac{1}{4}x^2\right) = \sum_{n=0}^{\infty} 2^{\frac{1}{2}n} \cos\frac{n\pi}{4} \cdot \frac{x^n}{n!}; \tag{1.8}$$

$$(1.9)\qquad e^x\, {}_0F_2\left(\frac{1}{3}, \frac{2}{3}; -\frac{x^3}{27}\right) = 1 + 2\sum_{n=1}^{\infty} 3^{\frac{1}{2}n-1} \cos\frac{n\pi}{6}\cdot\frac{x^n}{n!}.$$

Indeed, put

$$e^x\, {}_0F_2\left(\frac{1}{3}, \frac{2}{3}; -\frac{x^3}{27}\right) := 1 + \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}.$$

It is a routine work to get

$$a_n = \sum_{m=0}^{[n/3]} \frac{(-1)^m\, n!}{(n-3m)!\,\left(\frac{1}{3}\right)_m \left(\frac{2}{3}\right)_m m!\, 3^{3m}}.$$

Using (1.2) and (1.5), we obtain

$$a_n = {}_3F_2\left[-\frac{n}{3}, \frac{-n+1}{3}, \frac{-n+2}{3}; \frac{1}{3}, \frac{2}{3}; 1\right],$$

which, using Dixon's theorem (1.7), reduces to

$$a_n = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2^{-\frac{n}{3}}\Gamma\left(\frac{1}{2}-\frac{n}{6}\right)\Gamma\left(\frac{1}{3}+\frac{n}{6}\right)\Gamma\left(\frac{2}{3}+\frac{n}{6}\right)\Gamma\left(\frac{n}{3}\right)}.$$

Finally making use of (1.4) ($m = 2$ and $m = 3$) and (1.3) yields (1.9). Similarly, using Kummer's theorem (1.6) and (1.3), we can prove (1.8). On the other hand, Bailey also generalizes (1.8) and (1.9) by using an elementary method totally different from those given just above as follows (see [1, pp. 247-248, Eq. (3.8)]): For λ a prime,

$$(1.10)\qquad \begin{aligned} &e^x\, {}_0F_{\lambda-1}\left[\frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lambda-1}{\lambda}; -\left(\frac{x}{\lambda}\right)^{\lambda}\right] \\ &= 1 + \frac{1}{\lambda}\sum_{n=1}^{\infty}\left\{\sum_{r=1}^{\lambda} \cos^n\frac{(2r-1)\pi}{2\lambda}\cos\frac{n(2r-1)\pi}{2\lambda}\right\}\frac{2^n x^n}{n!}, \end{aligned}$$

which, for $\lambda = 2$ and $\lambda = 3$, reduces immediately to (1.8) and (1.9).

In this note we are also aiming at providing some product formulas involving the generalized hypergeometric series of similar nature of (1.10) by modifying the elementary method proposed by Bailey in the proof of (1.10). Indeed, we obtain the following presumably new formulas: For λ an odd prime,

(1.11)
$$(-1)^{\frac{\lambda+1}{2}} \frac{x^\lambda}{\lambda!}\, {}_0F_{2\lambda-1}\left[\frac{1}{2\lambda}+\frac{1}{2}, \frac{1}{2\lambda}+1, \frac{2}{2\lambda}+\frac{1}{2}, \frac{2}{2\lambda}+1, \cdots, \frac{\lambda-1}{2\lambda}+\frac{1}{2}, \frac{\lambda-1}{2\lambda}+1, \frac{3}{2}; \left(\frac{-x^2}{4\lambda^2}\right)^\lambda\right]$$
$$= \frac{1}{\lambda}\sum_{n=1}^{\infty}\left\{\sum_{r=1}^{\lambda}\cos^n\left(\frac{\pi}{4}+\frac{(2r-1)\pi}{2\lambda}\right)\sin\left(\frac{n\pi}{4}+\frac{(2r-1)n\pi}{2\lambda}\right)\right\}\frac{2^n}{n!}x^n$$

and

$$e^x\, {}_0F_{\lambda-1}\left[\frac{1}{\lambda}, \frac{2}{\lambda}, \cdots, \frac{\lambda-1}{\lambda}; \left(\frac{x}{\lambda}\right)^\lambda\right]$$
(1.12)
$$= 1+\frac{1}{\lambda}\sum_{n=1}^{\infty}\left\{\sum_{r=1}^{\lambda}\cos^n\frac{r\pi}{\lambda}\cos\left(\frac{rn\pi}{\lambda}\right)\right\}\frac{2^n}{n!}x^n,$$

where λ is a prime.

It should be remarked in passing that the formula (1.11) is seemingly a contiguous analogue of (1.10) and the formula (1.10) is concerned with an alternating series in comparison with (1.12).

2. Proofs of Formulas (1.11) and (1.12)

Let $w = e^{\frac{\pi i}{\lambda}}$, where λ is an odd prime. Consider the function

$$f(x) := \sum_{r=1}^{\lambda} e^{xiw^{2r-1}} := \sum_{n=0}^{\infty} a_n x^n,$$

where the coefficient a_n is, in terms of Maclaurin series expansion,

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (n \in \mathbf{N} \cup \{0\}).$$

We find that

$$f^{(n)}(0) = \frac{i^n}{w^n} \sum_{r=1}^{\lambda} w^{2rn}$$

and also observe that

$$\begin{aligned} \sum_{r=1}^{\lambda} w^{2rn} &= \sum_{r=1}^{\lambda} e^{\frac{2r\pi i}{\lambda} n} \\ &= \begin{cases} 0 & \text{if } n \text{ is not a multiple of } \lambda, \\ \lambda & \text{if } n \text{ is a multiple of } \lambda. \end{cases} \end{aligned}$$

From the above observation, we have

$$a_{\lambda n} = \frac{f^{(\lambda n)}(0)}{(\lambda n)!} = \frac{(-1)^n\, i^{\lambda n}}{(\lambda n)!} \lambda$$

and otherwise the coefficient a_n vanishes.

We thus have

(2.1)

$$\begin{aligned} \sum_{r=1}^{\lambda} e^{xiw^{2r-1}} &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^n\, i^{n\lambda}}{(n\lambda)!} x^{n\lambda} \\ &= \lambda \left\{ \sum_{m=0}^{\infty} \frac{i^{2m\lambda}}{(2m\lambda)!} x^{2m\lambda} - \sum_{m=0}^{\infty} \frac{i^{(2m+1)\lambda}}{((2m+1)\lambda)!} x^{(2m+1)\lambda} \right\} \\ &= \lambda\, {}_0F_{2\lambda-1} \left[\frac{1}{2\lambda}, \frac{1}{2\lambda} + \frac{1}{2}, \frac{2}{2\lambda}, \frac{2}{2\lambda} + \frac{1}{2}, \cdots, \right. \\ &\qquad \left. \frac{\lambda-1}{2\lambda}, \frac{\lambda-1}{2\lambda} + \frac{1}{2}, \frac{1}{2}; \left(\frac{-x^2}{4\lambda^2} \right)^{\lambda} \right] \\ &\quad + i(-1)^{\frac{\lambda+1}{2}} \frac{x^{\lambda}}{(\lambda-1)!}\, {}_0F_{2\lambda-1} \left[\frac{1}{2\lambda} + \frac{1}{2}, \frac{1}{2\lambda} + 1, \frac{2}{2\lambda} + \frac{1}{2}, \right. \\ &\qquad \left. \frac{2}{2\lambda} + 1, \cdots, \frac{\lambda-1}{2\lambda} + \frac{1}{2}, \frac{\lambda-1}{2\lambda} + 1, \frac{3}{2}; \left(\frac{-x^2}{4\lambda^2} \right)^{\lambda} \right]. \end{aligned}$$

Indeed, using (1.5), we have

$$\sum_{m=0}^{\infty}\frac{i^{2m\lambda}}{(2m\lambda)!}x^{2m\lambda}=\sum_{m=0}^{\infty}\frac{(-1)^{m\lambda}\left(x^{2\lambda}\right)^m}{(1)_{2m\lambda}}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m\lambda}\left(x^{2\lambda}\right)^m}{\lambda^{2m\lambda}\,2^{2m}\left(\frac{\frac{1}{\lambda}}{2}\right)_m\left(\frac{\frac{1}{\lambda}+1}{2}\right)_m 2^{2m}\left(\frac{\frac{2}{\lambda}}{2}\right)_m\left(\frac{\frac{2}{\lambda}+1}{2}\right)_m\cdots 2^{2m}\left(\frac{1}{2}\right)_m\left(\frac{2}{2}\right)_m}$$

$$={}_0F_{2\lambda-1}\left[\frac{1}{2\lambda},\frac{1}{2\lambda}+\frac{1}{2},\frac{2}{2\lambda},\frac{2}{2\lambda}+\frac{1}{2},\cdots,\right.$$
$$\left.\frac{\lambda-1}{2\lambda},\frac{\lambda-1}{2\lambda}+\frac{1}{2},\frac{1}{2},\frac{1}{2};\left(\frac{-x^2}{4\lambda^2}\right)^{\lambda}\right].$$

Next, λ being an odd prime and letting $\lambda=2l+1$,

$$\sum_{m=0}^{\infty}\frac{i^{(2m+1)\lambda}}{((2m+1)\lambda)!}x^{(2m+1)\lambda}=-(ix)^{\lambda}\sum_{m=0}^{\infty}\frac{(-1)^{m\lambda}\left(x^{2\lambda}\right)^m}{(1)_{(2m+1)\lambda}}$$

$$=i(-1)^{\frac{\lambda+1}{2}}\frac{x^{\lambda}}{\lambda!}\,{}_0F_{2\lambda-1}\left[\frac{1}{2\lambda}+\frac{1}{2},\frac{1}{2\lambda}+1,\frac{2}{2\lambda}+\frac{1}{2},\frac{2}{2\lambda}+1,\cdots,\right.$$
$$\left.\frac{\lambda-1}{2\lambda}+\frac{1}{2},\frac{\lambda-1}{2\lambda}+1,\frac{3}{2};\left(\frac{-x^2}{4\lambda^2}\right)^{\lambda}\right]$$

by considering the following identity

$$(1)_{(2m+1)\lambda}=\lambda^{(2m+1)\lambda}\prod_{j=1}^{\lambda}\left(\frac{j}{\lambda}\right)_{2m+1}=\lambda^{(2m+1)\lambda}\prod_{j=1}^{\lambda}\frac{j}{\lambda}\left(\frac{j}{\lambda}+1\right)_{2m}$$
$$=\lambda^{(2m+1)\lambda}\frac{2^{2m\lambda}\lambda!}{\lambda^{\lambda}}\prod_{j=1}^{\lambda}\left(\frac{j}{2\lambda}+\frac{1}{2}\right)_m\left(\frac{j}{2\lambda}+1\right)_m.$$

On the other hand, we have

$$e^{x}\sum_{r=1}^{\lambda}e^{xiw^{2r-1}}=\sum_{n=0}^{\infty}\sum_{r=1}^{\lambda}\frac{(1+iw^{2r-1})^n}{n!}x^n$$
$$=\lambda+\sum_{n=1}^{\infty}\sum_{r=1}^{\lambda}\frac{(1+iw^{2r-1})^n}{n!}x^n \tag{2.2}$$

and so, if $n \geq 1$,

$$\sum_{r=1}^{\lambda} \frac{(1+iw^{2r-1})^n}{n!}$$
$$= \frac{2^n}{n!} \sum_{r=1}^{\lambda} \cos^n\left(\frac{\pi}{4} + \frac{(2r-1)\pi}{2\lambda}\right)$$
$$\times \left[\cos\left(\frac{n\pi}{4} + \frac{(2r-1)n\pi}{2\lambda}\right) + i\sin\left(\frac{n\pi}{4} + \frac{(2r-1)n\pi}{2\lambda}\right)\right]$$

by noting

$$\begin{aligned}(1+iw^{2r-1})^n &= \left(1+e^{\left(\frac{1}{2}+\frac{2r-1}{\lambda}\right)\pi i}\right)^n \\ &= (1+\cos\pi\theta + i\sin\pi\theta)^n \\ &= 2^n \cos^n \frac{\pi\theta}{2}\left(\cos\frac{n\pi\theta}{2} + i\sin\frac{n\pi\theta}{2}\right),\end{aligned}$$

where $\theta = 1/2 + (2r-1)/\lambda$.

We therefore obtain the formula (1.11) by equating the imaginary part of (2.1) and (2.2). Moreover, comparing the real part of (2.1) and (2.2) yields another presumably new formula: For λ an odd prime,

(2.3)
$$e^x\,{}_0F_{2\lambda-1}\left[\frac{1}{2\lambda}, \frac{1}{2\lambda}+\frac{1}{2}, \frac{2}{2\lambda}, \frac{2}{2\lambda}+\frac{1}{2}, \cdots, \frac{\lambda-1}{2\lambda}, \frac{\lambda-1}{2\lambda}+\frac{1}{2}, \frac{1}{2}; \left(\frac{-x^2}{4\lambda^2}\right)^{\lambda}\right]$$
$$= 1 + \frac{1}{\lambda}\sum_{n=1}^{\infty}\left\{\sum_{r=1}^{\lambda}\cos^n\left(\frac{\pi}{4}+\frac{(2r-1)\pi}{2\lambda}\right)\cos\left(\frac{n\pi}{4}+\frac{(2r-1)n\pi}{2\lambda}\right)\right\}\frac{2^n}{n!}x^n.$$

Setting $\lambda = 3$ in (2.3), we obtain the following formula

(2.4)
$$e^x\,{}_0F_5\left[\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}; \left(\frac{-x^2}{36}\right)^3\right]$$
$$= 1 + \frac{1}{3}\sum_{n=1}^{\infty}\left\{\cos^n\frac{5\pi}{12}\cos\frac{5n\pi}{12} + \cos^n\frac{3\pi}{4}\cos\frac{3n\pi}{4} + \cos^n\frac{13\pi}{12}\cos\frac{13n\pi}{12}\right\}\frac{2^n}{n!}x^n.$$

We also note that the case $\lambda = 6$ in (1.10) seemingly agrees with the formula (2.4), but since λ is a prime, we cannot get the formula (2.4) from (1.10).

Similarly as in getting (1.11) and (2.3), considering the function

$$g(x) := \sum_{r=1}^{\lambda} e^{xw^{2r}} \quad (w = e^{\pi i/\lambda} \text{ and } \lambda \text{ a prime}),$$

we readily obtain the formula (1.12).

ACKNOWLEDGMENTS. The authors would like to take this opportunity to express our graditude for the referee's helpful comments. The first and second-named authors wish to acknowledge the financial support of the Korea Research Foundation made in the program year of 1998, Project No. 1998-015-D00022, and the third-named author was partially supported by the Basic Science Institute Program of the Ministry of Education of Korea (now the Korea Research Foundation) under Project BSRI-98-1431.

References

[1] W. N. Bailey, *Products of generalized hypergeometric series*, Proc. London Math. Soc. **28** (1928), no. 2, 242–254.

[2] A. C. Dixon, *Summation of a certain series*, Proc. London Math. Soc. **35** (1903), no. 1, 285-289.

[3] E. E. Kummer, *Über die hypergeometrische Reihe ...*, J. Reine Angew. Math. **15** (1836), 39–83 and 127–172.

[4] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

Y. J. Cho and T. Y. Seo
Department of Mathematics
College of Natural Sciences
Pusan National University
Pusan 609-735, Korea
E-mail: Tyseo@hyowon.pusan.ac.kr

Junesang Choi
Department of Mathematics
College of Natural Sciences
Dongguk University
Kyongju 780-714, Korea
E-mail: junesang@email.dongguk.ac.kr