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ON THE ALMOST SURE CONVERGENCE OF WEIGHTED SUMS OF 2-DIMENSIONAL ARRAYS OF POSITIVE DEPENDENT RANDOM VARIABLES

TAE-SUNG KIM, HO-YU BAEK AND KWANG-HEE HAN

ABSTRACT. In this paper we derive the almost sure convergence of weighted sums of 2-dimensional arrays of random variables which are either pairwise positive quadrant dependent or associated. Our results imply an extension of Etemadi's (1983) strong laws of large numbers for weighted sums of nonnegative random variables to the 2-dimensional case.

1. Introduction

Lehmann (1966) introduced the notion of positive quadrant dependence: A sequence $\{X_i : i \geq 1\}$ of random variables is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real r_i, r_j and $i \neq j$, $P\{X_i > r_i,\ X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}$ (or $P\{X_i \leq r_i,\ X_j \leq r_j\} \geq P\{X_i \leq r_i\}P\{X_j \leq r_j\}$. A much stronger concept than PQD was considered by Esary, Proschan, and Walkup (1967): A sequence $\{X_i : i \geq 1\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)}, \cdots, X_{j(n)}\}$ and any real coordinatewise nondecreasing functions f, g on R^n, $Cov(f(X_{j(1)}, \cdots, X_{j(n)}), g(X_{j(1)}, \cdots, X_{j(n)})) \geq 0$, whenever the covariance is defined. Let us remark that associated random variables are always pairwise PQD and that (pairwise) independent random variables are (pairwise PQD) associated. For a sequence

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of random variables the strong law of large numbers are investigated extensively in the literature. Etemadi (1981) derived the strong law of large numbers for a sequence of pairwise independent, identically distributed random variables and extended it to the d-dimensional array of random variables. Etemadi (1983 a) studied the strong law of large numbers for a sequence of nonnegative random variables which are pairwise negatively dependent and Etemadi (1983b) derived the almost sure convergence of weighted sums of nonnegative random variables.

Birkel (1989) also studied the strong laws of large numbers for sequences of random variables which are either pairwise positively quadrant dependent or associated. Kim et al. (1998) extended the Birkel's (1989) results to the 2-dimensional case (see [7]).

Let $\{X_{\underline{j}}, \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of random variables, assumed throughout this work to be nondegenerate, with finite first moment and let $\{w_{\underline{j}}, \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of positive numbers. Define $S_{\underline{n}} = \sum_{\underline{1} \leq \underline{j} \leq \underline{n}} w_{\underline{j}} X_{\underline{j}}$ and $W_{\underline{n}} = \sum_{\underline{1} \leq \underline{j} \leq \underline{n}} w_{\underline{j}}$.

In this paper we study the almost sure convergence of $(S_{\underline{n}} - ES_{\underline{n}})/W_{\underline{n}}$ to zero as $\underline{n} \to \infty$ for the case where the 2-dimensional positive dependent random variables are nonnegative and $\{w_{\underline{j}},\ \underline{j} \geq \underline{1}\}$ satisfy

$$(1) \qquad \frac{w_{\underline{n}}}{W_{\underline{n}}} \to 0 \text{ and } \mathrm{W}_{\underline{\mathrm{n}}} \to \infty \text{ as } \underline{\mathrm{n}} \to \infty,$$

where $\underline{n} \to \infty$ means that $|\underline{n}| = n_1 \times n_2 \to \infty$ for $\underline{n} = (n_1, n_2)$.

Then we apply the result to the 2-dimensional array of random variables which are either pairwise PQD or associated.

In Section 2 we obtain a strong law of large numbers of weighted sums of 2-dimensional arrays of positively dependent nonnegative random variables by extending Theorem 1 in Etemadi (1983 b). In Section 3 we derive strong laws of large numbers for 2-dimensional arrays of random variables which are either pairwise positive quadrant dependent or associated.

2. Preliminaries

By extending Theorem 1 of Etemadi (1983 b) we obtain the following theorem which will play the key role to derive our main results.

THEOREM 2.1. *Let $\{w_{\underline{j}}, \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of positive numbers satisfying* (1) *and let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array* *of positively dependent nonnegative random variables with finite second moments such that*

$$\sup_{\underline{j}\geq\underline{1}} EX_{\underline{j}} < \infty, \tag{2}$$

$$\sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} \frac{w_{\underline{i}}w_{\underline{j}}Cov(X_{\underline{i}}, X_{\underline{j}})}{W_{\underline{j}}^2} < \infty. \tag{3}$$

Let $S_{\underline{n}} = \sum_{\underline{1}\leq\underline{j}\leq\underline{n}} w_{\underline{j}}X_{\underline{j}}$. *Then as* $\underline{n} \to \infty, (S_{\underline{n}} - ES_{\underline{n}})/W_{\underline{n}} \to 0$ a.s., *where* $\underline{n} \to \infty$ *means that* $|\underline{n}| = n_1 \times n_2 \to \infty$ *for* $\underline{n} = (n_1, n_2)$.

Proof. Let $a > 1$, $b > 1$ and for each $\underline{k} = (k_1, k_2)$, set $n_{\underline{k}} = \underline{n}$ such that $n_{\underline{k}} \leq \underline{n} < n_{\underline{k}+\underline{1}}$ and $a^{k_1}b^{k_2} \leq W_{\underline{n}} < a^{k_1+1}b^{k_2+1}$. Since $W_{\underline{n}}/W_{\underline{n}+\underline{1}} \to 1$ as $\underline{n} \to \infty$, it follows that $W_{n_{\underline{k}}} \sim a^{k_1}b^{k_2}$ for all large k_1 and k_2. Therefore for some positive c and every $\underline{i} \geq \underline{1}$, $\{\underline{k} : n_{\underline{k}} \geq \underline{i}\} \subset \{\underline{k} : W_{n_{\underline{k}}} > W_{\underline{i}}\} \subset \{\underline{k} : ca^{k_1}b^{k_2} \geq W_{\underline{i}}\}$ and thus by Chebyshev's inequality we have for every $\epsilon > 0$

$$\begin{aligned} \sum_{\underline{k}\geq\underline{1}} P\{|S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}| \geq \epsilon|W_{n_{\underline{k}}}|\} &\leq c_1 \sum_{\underline{k}\geq\underline{1}} \frac{Var(S_{n_{\underline{k}}})}{W_{n_{\underline{k}}}^2} \\ &= c_1 \sum_{\underline{k}\geq\underline{1}} \sum_{\underline{1}\leq\underline{j}\leq n_{\underline{k}}} \sum_{\underline{1}\leq\underline{i}\leq n_{\underline{k}}} \frac{w_{\underline{i}}w_{\underline{j}}Cov(X_{\underline{i}}, X_{\underline{j}})}{W_{n_{\underline{k}}}^2} \\ &\leq c_2 \sum_{\underline{k}\geq\underline{1}} \sum_{\underline{1}\leq\underline{j}\leq n_{\underline{k}}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} \frac{w_{\underline{i}}w_{\underline{j}}Cov(X_{\underline{i}}, X_{\underline{j}})}{a^{2k_1}b^{2k_2}} \\ &\leq c_3 \sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} \frac{w_{\underline{i}}w_{\underline{j}}Cov(X_{\underline{i}}, X_{\underline{j}})}{W_{\underline{j}}^2} < \infty \end{aligned} \tag{4}$$

which yields by the Borel-Cantelli lemma

$$\frac{S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}}{W_{n_{\underline{k}}}} \to 0 \text{ a.s.} \tag{5}$$

Now given $\underline{k} = (k_1, k_2)$, positive integers k_1, k_2 for $n_{\underline{k}} \leq \underline{n} < n_{\underline{k}+\underline{1}}$ we have

$$\left|\frac{S_{\underline{n}} - ES_{\underline{n}}}{W_{\underline{n}}}\right| \leq \left|\frac{S_{n_{\underline{k}-\underline{1}}} - ES_{n_{\underline{k}-\underline{1}}}}{W_{n_{\underline{k}-\underline{1}}}}\right| \frac{W_{n_{\underline{k}-\underline{1}}}}{W_{n_{\underline{k}}}} + \frac{ES_{n_{\underline{k}-\underline{1}}} - ES_{n_{\underline{k}}}}{W_{n_{\underline{k}}}} \tag{6}$$

by using the monotonicity of $S_{\underline{n}}$. Note that the first term on the right-hand side of (6) converges almost surely to zero by (5). From (2) and the fact that $W_{n_{\underline{k}}} \sim a^{k_1}b^{k_2}$ the second term on the right-hand side of (6) becomes

$$\frac{ES_{n_{\underline{k+1}}} - ES_{n_{\underline{k}}}}{W_{n_{\underline{k}}}} \leq \sup_{\underline{i}\geq\underline{1}} EX_{\underline{i}}(ab-1) \tag{7}$$

and thus

$$\limsup_{\underline{n}} \left(\frac{|S_{\underline{n}} - ES_{\underline{n}}|}{W_{\underline{n}}}\right) \leq \sup_{\underline{i}\geq\underline{1}} EX_{\underline{i}}(ab-1) \tag{8}$$

for every $a > 1$ and $b > 1$ which concludes the proof since a and b may be arbitrary close to 1. □

COROLLAY 2.1. *Let $\{w_{\underline{j}}, \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of positive numbers and let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of pairwise independent random variables with finite second moments such that*

$$\sup_{\underline{j}\geq\underline{1}} E|X_{\underline{j}} - EX_{\underline{j}}| < \infty, \tag{9}$$

$$\sum_{\underline{j}\geq\underline{1}} \frac{w_{\underline{j}}^2 Var(X_{\underline{j}})}{W_{\underline{j}}^2} < \infty. \tag{10}$$

Let $S_{\underline{n}} = \sum_{\underline{1}\leq\underline{j}\leq\underline{n}} w_{\underline{j}}X_{\underline{j}}$. Then as $\underline{n} \to \infty$, $(S_{\underline{n}} - ES_{\underline{n}})/W_{\underline{n}} \to 0$ a.s.

PROOF. Let $S_{\underline{n}}^* = \sum_{\underline{1}\leq\underline{j}\leq\underline{n}} w_{\underline{j}}(X_{\underline{j}} - EX_{\underline{j}})^+$ and its negative counter part, $S_{\underline{n}}^{**} = \sum_{\underline{i}\leq\underline{j}\leq\underline{n}} w_{\underline{j}}(X_{\underline{j}} - EX_{\underline{j}})^-$. Then from Theorem 2.1 and the fact that for $\underline{j} \geq \underline{1}$ $Var[w_{\underline{j}}(X_{\underline{j}} - EX_{\underline{j}})^+] \leq E[w_{\underline{j}}^2(X_{\underline{j}} - EX_{\underline{j}})^{+2}] \leq Var[w_{\underline{j}}X_{\underline{j}}]$ we obtain the desired result by noting that $ES_{\underline{n}}^* - ES_{\underline{n}}^{**} = 0$. □

Finally we introduce the following maximal inequality for 2-dimensional array of associated random variables in Newman and Wright (1982).

LEMMA 2.1. (Newman, Wright, 1982) *Let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of associated random variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Then, for $\lambda_2 > \lambda_1 \geq 0$, we have*

$$P\{\max_{\underline{1}\leq\underline{i}\leq\underline{n}} S_{\underline{i}} \geq \lambda_2\} \leq 3^{\frac{3}{2}}2^{-1}\left(\frac{ES_{\underline{n}}^2}{(\lambda_2-\lambda_1)^2}\right)^{\frac{3}{4}}[P(S_{\underline{n}} \geq \lambda_1)]^{\frac{1}{4}}. \tag{11}$$

PROOF. See the proof of Theorem 10 of Newman and Wright (1982). □

3. Main Results

The following theorem is the almost sure convergence of weighted sums of 2-dimensional arrays of PQD random variables.

THEOREM 3.1. *Let* $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ *be a 2-dimensional array of pairwise PQD random variables with finite variance. Assume*

$$\sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} \frac{w_{\underline{i}} w_{\underline{j}} Cov(X_{\underline{i}}, X_{\underline{j}})}{W_{\underline{j}}^2} < \infty, \tag{12}$$

$$\sup_{\underline{j}\geq\underline{1}} E|X_{\underline{j}} - EX_{\underline{j}}| < \infty. \tag{13}$$

Then, as $\underline{n} \to \infty, (S_{\underline{n}} - ES_{\underline{n}})/W_{\underline{n}} \to 0$ *a.s., where* $S_{\underline{n}} = \sum_{\underline{1}\leq\underline{j}\leq\underline{n}} w_{\underline{j}} X_{\underline{j}}$.

PROOF. First note that $(X_{\underline{i}}, X_{\underline{j}})$ is PQD if and only if

$$Cov(f(X_{\underline{i}}), g(X_{\underline{j}})) \geq 0,$$

for all nondecreasing (nonincreasing) functions f, g such that the covariance exists (see Lemma 1 of Lehmann (1966)). Hence the $w_{\underline{j}}(X_{\underline{j}} - EX_{\underline{j}})$, $\underline{j} \geq \underline{1}$, are pairwise PQD and without loss of generality, we may assume that, for $\underline{j} \geq \underline{1}, EX_{\underline{j}} = 0$. Now we consider $\{X_{\underline{j}}^+ : \underline{j} \geq \underline{1}\}$ and its corresponding sum, say $S_{\underline{n}}^* = \sum_{\underline{1}\leq\underline{i}\leq\underline{n}} w_{\underline{i}} X_{\underline{i}}^+$. Our assumptions (12) and (13) together with Theorem 2.1 imply that, as $\underline{n} \to \infty, (S_{\underline{n}}^* - ES_{\underline{n}}^*)/W_{\underline{n}} \to 0$ a.s. A similar consideration for the negative parts, say $S_{\underline{n}}^{**} = \sum_{\underline{1}\leq\underline{i}\leq\underline{n}} w_{\underline{i}} X_{\underline{i}}^-$, and the fact that $ES_{\underline{n}}^* - ES_{\underline{n}}^{**} = 0$ complete the proof of Theorem 3.1. □

If the random variables are associated, assumption (13) in Theorem 3.1 may be dropped and we need maximal inequality to prove the following theorem:

THEOREM 3.2. *Let* $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ *be a 2-dimensional array of associated random variables with finite variance. If* (12) *holds then, as* $\underline{n} \to \infty, (S_{\underline{n}} - ES_{\underline{n}})/W_{\underline{n}} \to 0$ *a.s.*

PROOF. Since the random variables $w_{\underline{j}}(X_{\underline{j}} - EX_{\underline{j}})', \underline{j} \geq \underline{1}$ are associated by (P_4) of Esary, Proschan and Walkup (1967), without loss of generality we may assume that, for $\underline{j} \geq \underline{1}, EX_{\underline{j}} = 0$. We may also assume that

$X_{\underline{j}} \geq 0$. As in the proof of Theorems 2.1 by Chebyshev's inequality for every $\epsilon > 0$,

$$(14) \sum_{\underline{k}\geq \underline{1}} P\Big(\frac{|S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}|}{|W_{n_{\underline{k}}}|} > \epsilon\Big) \leq c \sum_{\underline{j}\geq \underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} \frac{w_{\underline{i}} w_{\underline{j}} Cov(X_{\underline{i}}, X_{\underline{j}})}{W_{\underline{j}}^2} < \infty,$$

where c is an unimportant constant. Thus by the Borel-Cantelli lemma, as $n_{\underline{k}} \to \infty$

$$(15) \qquad \frac{S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}}{W_{n_{\underline{k}}}} \to 0 \text{ a.s.}$$

By standard argument, it suffices to show that, as $n_{\underline{k}} \to \infty$,

$$(16) \qquad \max_{n_{\underline{k}} < \underline{i} \leq n_{\underline{k}+\underline{1}}} \frac{|S_{\underline{i}} - S_{n_{\underline{k}}}|}{W_{n_{\underline{k}}}} \to 0 \text{ a.s.}$$

Let $a > 1, b > 1$ and let for each $k = (k_1, k_2), \lambda_2 = \epsilon W_{n_{\underline{k}}}$ and $\lambda_1 = \frac{1}{2}\epsilon W_{n_{\underline{k}}}$. Using Chebyshev's inequality and applying (11) of Lemma 2.1 we have

$$\begin{aligned}
& P\Big\{ \max_{n_{\underline{k}} < \underline{i} \leq n_{\underline{k}+\underline{1}}} \frac{S_{\underline{i}} - S_{n_{\underline{k}}}}{W_{n_{\underline{k}}}} \geq \epsilon \Big\} \\
\leq \;& 3^{\frac{3}{2}} 2^{-1} \Big(\frac{E(S_{n_{\underline{k}+\underline{1}}} - S_{n_{\underline{k}}})^2}{(\frac{1}{2}\epsilon W_{n_{\underline{k}}})^2}\Big)^{\frac{3}{4}} \Big(\frac{E(S_{n_{\underline{k}+\underline{1}}} - S_{n_{\underline{k}}})^2}{(\frac{1}{2}\epsilon W_{n_{\underline{k}}})^2}\Big)^{\frac{1}{4}} \\
(17) \qquad \leq \;& d_1 W_{n_{\underline{k}}}^{-2} Var(S_{n_{\underline{k}+\underline{1}}} - S_{n_{\underline{k}}}) \\
\leq \;& d_1 W_{n_{\underline{k}}}^{-2} Var(S_{n_{\underline{k}+\underline{1}}}) \\
\leq \;& a^2 b^2 d_2 W_{n_{\underline{k}+\underline{1}}}^{-2} Var(S_{n_{\underline{k}-\underline{1}}}), \quad \text{where } \; d_1 > 0, \; d_2 > 0,
\end{aligned}$$

since the $w_{\underline{j}} X_{\underline{j}}$ are nonnegatively correlated. Replacing the random variables $w_{\underline{j}} X_{\underline{j}}$ by their negatives (which are also associated according to (P_4) of Esary, Proschan and Walkup (1967)) we get the analogous inequality

$$(18) \quad P\{W_{n_{\underline{k}}}^{-1} \max_{n_{\underline{k}} < \underline{i} \leq n_{\underline{k}-\underline{1}}} -(S_{\underline{i}} - S_{n_{\underline{k}}}) \geq \epsilon\} \leq a^2 b^2 d_2 W_{n_{\underline{k}-\underline{1}}}^{-2} Var(S_{n_{\underline{k}-\underline{1}}}).$$

Consequently (17) and (18) imply

$$(19) \quad P\{W_{n_{\underline{k}}}^{-1} \max_{n_{\underline{k}} < \underline{i} \leq n_{\underline{k}-\underline{1}}} |S_{\underline{i}} - S_{n_{\underline{k}}}| \geq \epsilon\} \leq 2a^2 b^2 d_2 W_{n_{\underline{k}-\underline{1}}}^{-2} Var(S_{n_{\underline{k}-\underline{1}}})$$

and hence

$$\begin{aligned}\sum_{\underline{k}\geq\underline{1}} P\Big\{ & W_{n_{\underline{k}}}^{-1} \max_{n_{\underline{k}}<\underline{i}\leq n_{\underline{k}-\underline{1}}} |S_{\underline{i}} - S_{n_{\underline{k}}}| \geq \epsilon \Big\} \\ & \leq 2a^2b^2d_2 \sum_{\underline{k}\geq\underline{1}} W_{n_{\underline{k}-\underline{1}}}^{-2} Var(S_{n_{\underline{k}-\underline{1}}}) \\ & \leq 2a^2b^2d_2 \sum_{\underline{k}\geq\underline{1}} W_{n_{\underline{k}}}^{-2} Var(S_{n_{\underline{k}}}) \\ & \leq 2a^2b^2d_3 \sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} \frac{w_{\underline{i}}w_{\underline{j}}Cov(X_{\underline{i}}, X_{\underline{j}})}{W_{\underline{j}}^2} \\ & < \infty\end{aligned}$$

according to (4), where d_2, d_3 are positive constants. Again applying the Borel-Cantelli lemma, we obtain (14) and complete the proof. □

References

[1] Birkel, T., *A note on the strong law of large numbers for positively dependent random variables*, Statist. Probab. Lett. **7** (1989), 17-20.

[2] Esary, J., Proschan, F. and Walkup, D., *Association of random variables with applications*, Ann. Math. Statist. **38** (1967), 1466-1474.

[3] Etemadi, N., *An elementary proof of the strong law of large numbers*, Z. Wahrsch. verw. Gebiete **55** (1981), 119-122.

[4] ———, *On the laws of large numbers for nonnegative random variables*, J. Multi. Anal. **13** (1983 a), 187-193.

[5] ———, *Stability of sums of weighted nonnegative random variables*, J. Multi. Anal. **13** (1983 b), 361-365.

[6] Lehmann, E. L., *Some concepts of dependence*, Ann. Math. Statist. **37** (1966), 1137-1153.

[7] Kim, T. S., Beak, H. Y. and Seo, H. Y., *On the strong laws of large numbers for 2-dimensional positively dependent random variables*, Bull. Korean Math. Soc. **35** (1998), 709-718.

[8] Newman, C. M. and Wright, A. L., *Associated random variables and martingale inequalities*, Z. Wahrsch. verw. Geb. **59** (1982), 361-371.

Tae-Sung Kim and Ho-Yu Baek
Division of Mathematical Science
Wonkwang University
Ik-San City, Chonbuk 570-749, Korea
E-mail: starkim@wonnms.wonkwang.ac.kr
hybaek@wonnms.wonkwang.ac.kr

Kwang-Hee Han
Division of Computer Science
Howon University
Kun-San City, Chonbuk 573-930, Korea
E-mail: khhan@sunny.howon.ac.kr