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THE OPTIMAL BIVARIATE BONFERRONI-TYPE LOWER BOUNDS

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ABSTRACT. Let $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, be the number of those A_i and B_j which occur. We establish bivariate lower bounds on the distribution $P(X \geq 1,\ Y \geq 1)$ and $P(X \geq i,\ Y \geq j)$ by linear combinations of the binomial moments $S_{i,j},\ 1 \leq i < m,\ 1 \leq j < n$, which extend and refine bivariate Bonferroni-type lower bounds given by Chen and Seneta (1995) and Lee (1997).

1. Introduction

Let $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, be the number of those A_i and B_j which occur. For integers $i \geq 0$ and $j \geq 0$, set

$$S_{i,j} = S_{i,j}(A,B) = E\left[\binom{X}{i}\binom{Y}{j}\right]. \tag{1}$$

For $i = 0$ or $j = 0$ we also write $S_{i,0}(A) = S_i(A)$ and $S_{0,j}(B) = S_j(B)$ and we know that $S_{0,0} = S_{0,0}(A,B) = 1$. The sets $S_{i,j}, S_i(A)$ and $S_j(B)$ are called the binomial moments of the vector (X,Y) and the variables X and Y, respectively.

By turning to indicator variables we immediately get that for $1 \leq i \leq m,\ 1 \leq j \leq n$,

$$S_{i,j} = \sum P(A_{r_1} \cap \cdots \cap A_{r_i} \cap B_{t_1} \cap \cdots \cap B_{t_j}) \tag{2}$$

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where the summation $\sum$ is over all subscripts satisfying $1 \leq r_1 < \cdots < r_i \leq m$ and $1 \leq t_1 < \cdots < t_j \leq n$.

Galambos and Xu (1993) established optimal bivariate lower bound on $P(X \geq 1,\ Y \geq 1)$ by means of linear combinations of the binomial moments $S_{i,j},\ 1 \leq i \leq 2,\ 1 \leq j \leq 2$, which those are generalized by Chen and Seneta (1995) later.

In this paper, we establish new inequalities which refine recent bivariate lower bounds given by Chen and Seneta (1995) and Lee (1997).

2. The Results

We establish sharper bivariate lower bounds on the distribution $P(X \geq 1,\ Y \geq 1)$ and $P(X \geq i,\ Y \geq j)$, by means of the bivariate binomial moments $S_{i,j},\ 1 \leq i \leq m,\ 1 \leq j \leq n$ and $S_{i,j},\ S_{i+1,j},\ S_{i,j+1},\ S_{i+1,j+1}$, respectively.

THEOREM 1. *For any positive integers r_1, r_2, a and b with $2r_1 \leq a < m,\ 2r_2 \leq b < n$,*

$$(3) \qquad P(X \geq 1,\ Y \geq 1) \geq \frac{1}{k}\left\{\sum_{i=1}^{2r_1}\sum_{j=1}^{2r_2}(-1)^{i+j}\binom{2r_1}{i}\binom{2r_2}{j}\frac{S_{i,j}}{\binom{a}{i}\binom{b}{j}}\right\}$$

where $k = \max\left\{1, F(m)G(n)\right\}$, where $F(m) = 1 - \frac{\binom{m-a+2r_1-1}{2r_1}}{\binom{a}{2r_1}}$ and $G(n) = 1 - \frac{\binom{n-b+2r_2-1}{2r_2}}{\binom{b}{2r_2}}$.

REMARK.

1) If $F(m) \geq 0$ or $G(n) \geq 0$, then $k = 1$.

2) If $r_1 = r_2 = 1$, then this inequality is same as that of Chen and Seneta (1995);

that is, $P(X \geq 1, Y \geq 1) \geq \frac{1}{k}\left\{\frac{4S_{1,1}}{ab} - \frac{2S_{1,2}}{a\binom{b}{2}} - \frac{2S_{2,1}}{\binom{a}{2}b} + \frac{S_{2,2}}{\binom{a}{2}\binom{b}{2}}\right\}$

where $k = \max\left\{1, \frac{mn(2a-1-m)(2b-1-n)}{a(a-1)b(b-1)}\right\}$.

3) This inequality is a refinement of Theorem 4 in Lee (1997).

THEOREM 2. *For any positive integers a, b, i and j with $a \geq 2,\ b \geq 2,\ i \geq 1,\ j \geq 1$,*

$$P(X \geq i,\ Y \geq j) \geq \frac{1}{k}\left\{ \frac{(i+1)(j+1)}{\binom{ai}{i}\binom{bj}{j}} S_{i,j} - \frac{(i+1)j}{\binom{ai}{i}\binom{bj}{j+1}} S_{i,j+1} \right.$$

$$\left. - \frac{i(j+1)}{\binom{ai}{i+1}\binom{bj}{j}} S_{i+1,j} + \frac{ij}{\binom{ai}{i+1}\binom{bj}{j+1}} S_{i+1,j+1} \right\} \tag{4}$$

where $k = \max\left\{1, \frac{\binom{m}{i}(ai+a-1-m)\binom{n}{j}(bj+b-1-n)}{\binom{ai}{i}(a-1)\binom{bj}{j}(b-1)}\right\}$.

This inequality is a generalization of that of Chen and Seneta (1995) and a refinement of Theorem 5 in Lee [1997]. If $i = j = 1$, then this inequality is same as that of Chen and Seneta (1995); that is,

$$P(X \geq 1,\ Y \geq 1) \geq \frac{1}{k}\left\{ \frac{4S_{1,1}}{ab} - \frac{2S_{1,2}}{a\binom{b}{2}} - \frac{2S_{2,1}}{\binom{a}{2}b} + \frac{S_{2,2}}{\binom{a}{2}\binom{b}{2}} \right\}$$

where $k = \max\left\{1, \frac{mn(2a-1-m)(2b-1-n)}{a(a-1)b(b-1)}\right\}$.

3. Proofs

The proofs of Theorems 1 and 2 are based on the method of indicators and are utilized the inequalities in Tan and Xu (1989).

PROOF. we use the inequality of Tan and Xu (1989); that is, for positive integers a, b, r_1 and r_2 with $2r_1 \leq a < m,\ 2r_2 \leq b < n$,

$$P(X \geq 1) \geq \sum_{i=1}^{2r_1} (-1)^{i+1} \binom{2r_1}{i} \frac{S_i}{\binom{a}{i}} \tag{5a}$$

$$P(Y \geq 1) \geq \sum_{j=1}^{2r_2} (-1)^{j+1} \binom{2r_2}{j} \frac{S_j}{\binom{b}{j}}. \tag{6a}$$

Turning to indicators, (5a) and (6a) become

$$\text{(5b)} \qquad I(X \geq 1) \geq \sum_{i=1}^{2r_1}(-1)^{i+1}\binom{2r_1}{i}\frac{\binom{x}{i}}{\binom{a}{i}}$$

$$\text{(6b)} \qquad I(Y \geq 1) \geq \sum_{j=1}^{2r_2}(-1)^{j+1}\binom{2r_2}{j}\frac{\binom{y}{j}}{\binom{b}{j}}.$$

Let the right hand side of $(5b) = F(x)$ and the right hand side of $(6b) = G(x)$. Then $F(x) = 1 - \frac{\binom{x-a+2r_1-1}{2r_1}}{\binom{a}{2r_1}}$ and $G(y) = 1 - \frac{\binom{y-b+2r_2-1}{2r_2}}{\binom{b}{2r_2}}$ for $2r_1 \leq a < m$, $2r_2 \leq b < n$ and both $F(x)$ and $G(y)$ are less than or equal 1, respectively.

We can multiply (5b) and (6b) without changing the inequalities if $F(x) \geq 0$ or $G(y) \geq 0$. But if both of them have negative values, $F(m)G(n)$ can be greater than 1 because both $F(x)$ and $G(y)$ are decreasing functions and have negative minimum values at $x = m$, $y = n$ for $x \geq 2a$, $y \geq 2b$, respectively.

Hence we can choose k such that $F(x)G(y) \leq k$ for all $1 \leq x \leq m$, $1 \leq y \leq n$, where $k = \max(1, F(m)G(n))$. Note that $k = 1$ if $F(m) \geq 0$ or $G(n) \geq 0$.

Upon dividing inequality of above inequality by k and using the method of indicators and taking expectations, we get (3). □

PROOF. We also use the inequality of Tan and Xu (1989); that is, for positive integers a, b, i and j with $i \geq 1$, $j \geq 1$, $a \geq 2$, $b \geq 2$,

$$\text{(7a)} \qquad P(X \geq i) \geq \frac{i+1}{\binom{ai}{i}}S_i - \frac{i}{\binom{ai}{i+1}}S_{i+1}$$

$$\text{(8a)} \qquad P(Y \geq j) \geq \frac{j+1}{\binom{bj}{j}}S_j - \frac{j}{\binom{bj}{j+1}}S_{j+1}.$$

Turning to indicators, (7a) and (8a) become

$$\text{(7b)} \quad I(X \geq i) \geq \frac{i+1}{\binom{ai}{i}}\binom{x}{i} - \frac{i}{\binom{ai}{i+1}}\binom{x}{i+1} = \frac{\binom{x}{i}(ai+a-1-x)}{\binom{ai}{i}(a-1)}$$

$$\text{(8b)} \quad I(Y \geq j) \geq \frac{j+1}{\binom{bj}{j}}\binom{y}{j} - \frac{j}{\binom{bj}{j+1}}\binom{y}{j+1} = \frac{\binom{y}{j}(bj+b-1-y)}{\binom{bj}{j}(b-1)}$$

Let the right hand side of $(7b) = H(x)$ and the right hand side of $(8b) = K(y)$. Then $H(x)$ and $K(y)$ have zeroes at $x = 0, 1, 2, \cdots i-1,\ ai+a-1$ and $y = 0, 1, 2, \cdots j-1,\ bj+b-1$, respectively, and are decreasing functions for $x > ai+a-1$ and $y > bj+b-1$, respectively.

Hence $H(x)$ and $K(y)$ have maximum value 1 at $x = ai-1,\ ai,\ y = bj-1,\ bj$, respectively. Also, they have negative minimum values at $x = m,\ y = n$ if $ai+a-1-x < 0,\ bj+b-1-y < 0$, respectively.

Therefore we can multiply $(7b)$ and $(8b)$ without changing the inequality if $H(x) \geq 0$ or $K(y) \geq 0$. But if both $H(m)$ and $K(n)$ are negative values, then $H(m)K(n)$ can be greater than 1.

Hence we can choose k such that $H(x)K(y) \leq k$ for all $1 \leq x \leq m,\ 1 \leq y \leq n$, where $k = \max(1, H(m)K(n))$. Note that $k = 1$ if $ai+a-1-m \geq 0$ or $bj+b-1-n \geq 0$.

Upon dividing inequality of above inequality by k and using the method of indicators and taking expectation, we get (4). □

4. Numerical Example

Let a machine consist of two pieces of equipment A and B. Let X_i be the time to failure of the i-th component of equipment A and let Y_j be the time to failure of the j-th component of equipment B. Assume that each X_i and each Y_j is unit exponential variates; that is, for each i, j;

$$P(X_i < x) = 1 - e^{-x}, \quad x > 0$$

and

$$P(Y_j < y) = 1 - e^{-y}, \quad y > 0$$

Consider a group A of ten components and a group B of three components. Let $X_1, X_2, \cdots, X_{10}$ be independent and identically distributed random variables and let Y_1, Y_2, Y_3 be independent and identically distributed random variables. We assume the structure is such that each X_i is completely dependent on each Y_j and it has probability zero that at least one component of equipment $A(B)$ fails within $x(y)$ period of time and all components of equipment $B(A)$ fail after $y(x)$ period of time; that is, for each $1 \le i \le 10$, $1 \le j \le 3$,

$$P\big(\cup_{i=1}^{10}(X_i < x)\ ,\ \cap_{j=1}^{3}(Y_j \ge y)\big) = P\big(\cap_{i=1}^{10}(X_i \ge x)\ ,\ \cup_{j=1}^{3}(Y_j < y)\big) = 0$$

We also specify the bivariate distributions and the trivariate distributions of the combination of X_i and Y_j. For simplicity, let us use the same bivariate and trivariate distribution for all dependent components. Let, for $1 \le i \le 10$, $1 \le j \le 3$,

$$P(X_i < x\ ,\ Y_j < y) = (1-e^{-x})(1-e^{-y})(1-\frac{1}{2}e^{-x-y}),$$
$$P(X_{i_1} < x\ ,\ X_{i_2} < x\ ,\ Y_j < y) = (1-e^{-x})^2(1-e^{-y})(1-\frac{1}{3}e^{-2x-y})$$

and

$$P(X_i < x\ ,\ Y_{j_1} < y\ ,\ Y_{j_2} < y) = (1-e^{-x})(1-e^{-y})^2(1-\frac{1}{3}e^{-x-2y}),$$

We can now compute $S_{1,1}$, $S_{1,2}$ and $S_{2,1}$. We have

$$S_{1,1} = \binom{10}{1}\binom{3}{1}(1-e^{-x})(1-e^{-y})(1-\frac{1}{2}e^{-x-y}),$$
$$S_{1,2} = \binom{10}{1}\binom{3}{2}(1-e^{-x})(1-e^{-y})^2(1-\frac{1}{3}e^{-x-2y}),$$
$$S_{2,1} = \binom{10}{2}\binom{3}{1}(1-e^{-x})^2(1-e^{-y})(1-\frac{1}{3}e^{-2x-y}),$$

Let us use the same 4-th multivariate distribution for all dependent components. Let

$$P(X_{i_1} < x,\ \ X_{i_2} < x,\ \ Y_{j_1} < y,\ Y_{j_2} < y)$$
$$= (1-e^{-x})^2(1-e^{-y})^2(1-\frac{1}{4}e^{-2x-2y}).$$

Also, we can now compute $S_{2,2}$. For a numerical calculation, let us choose $x = 0.1$ and $y = 0.2$. Let V_{10} be the number of those $A_i = (X_i < 0.1)$ which occur and let U_3 be the number of those $C_j = (Y_j < 0.2)$ which occur. By inequality of E. Galambos [1965] and Meyer [1969]; that is,

$$y_{1,1} = P(V_{10} \geq 1\ ,\ U_3 \geq 1) \geq S_{1,1} - S_{1,2} - S_{2,1}$$

we get $P(V_{10} \geq 1\ ,\ \ U_3 \geq 1) \geq 0.079$. But, when we use the inequalities of (2) in Remark, we get the sharper bounds

$$P(V_{10} \geq 1\ ,\ U_3 \geq 1) \geq 0.113$$

where we substitute $k = 1$, $a = 2$ and $b = 2$.

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