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SUMS OF CERTAIN CLASSES OF SERIES

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ABSTRACT. The object of this note is to give sums of certain families of series which are initiated from their special cases considered here. Relevant connections of the series identities presented here with those given elsewhere are also pointed out.

1. Introduction and Preliminaries

Vowe and Seiffert [6] proved the following series identity:

$$\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)}=\frac{2^n(n-1)!n!}{(2n)!}-\frac{1}{n\cdot 2^n} \tag{1.1}$$

$$(n \in \mathbf{N} := \{1,\, 2,\, 3,\, \ldots\})$$

by identifying the sum with the Eulerian integral

$$\int_0^1\left(1-\frac{t}{2}\right)^{n-1} t^n\, dt. \tag{1.2}$$

Srivastava [5] evaluated (1.1) as a special case by deducing a more general form of series identity with the aid of a summation formula involved in the hypergeometric series ${}_2F_1$ which is due to Kummer [2] (see also Rainville [4, p. 69, Exercise 3]):

$$ {}_2F_1\left(a,\, 1-a\,;\, b\,;\, \frac{1}{2}\right)=\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b+a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)} \quad (b \neq 0,\, -1,\, -2, \ldots), \tag{1.3}$$

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where the so-called hypergeometric series ${}_2F_1$ (also denoted by F) is defined by

$$(1.4)\qquad {}_2F_1\,(a,\,b\,;\,c\,;\,z) = {}_2F_1\left[\begin{matrix} a, & b\,; \\ & c\,; \end{matrix}\; z\right] := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

where a, b and c are arbitrary complex constants and $(\alpha)_n$ denotes the Pochhammer symbol (or the generalized factorial, since $(1)_n = n!$) defined by

$$(1.5)\qquad (\alpha)_n := \begin{cases} 1 & (n=0) \\ \alpha(\alpha+1)\ldots(\alpha+n-1) & (n \in \mathbf{N}). \end{cases}$$

From the fundamental functional relation of the Gamma function Γ, $\Gamma(z+1) = z\Gamma(z)$, we have

$$(1.6)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$(1.7)\qquad \{\Gamma(z)\}^{-1} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}},$$

γ being the Euler-Mascheroni's constant defined by

$$(1.8)\qquad \gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577\,215\,664\cdots.$$

From definitions (1.5) and (1.6), we can easily deduce the following formula:

$$(1.9)\qquad (\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k},$$

which, for the case $\alpha = 1$, yields immediately

$$(1.10)\qquad (-n)_k = \begin{cases} \dfrac{(-1)^k n!}{(n-k)!} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n. \end{cases}$$

In this note we are aiming at proving the following formulas contiguous to (1.1):

$$(1.11)\qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k}{2^k (n+k)(n+k+1)} = \frac{2^{n-1}(n!)^2}{(2n+1)!} - \frac{1}{2^{n+1}};$$

(1.12)

$$\sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{k}{2^k (n+k)(n+k+1)} = \frac{3 \cdot 2^n \cdot (n!)^2}{(n-1)\,(2n)!} - \frac{n+2}{(n-1)\,2^{n-1}}.$$

In fact, we are trying to give more general series identities including (1.11) and (1.12) as special cases by making use of known (or presumably new) summation formulas for ${}_2F_1$. We also point out relevant connections of the series identities presented here with those given elsewhere.

2. Various Series Identities

For simplicity in printing, we use the notations

$$\begin{aligned} F &= {}_2F_1(a, b; c; z), \\ F(a+) &= F(a+1, b; c; z), \\ F(a-) &= F(a-1, b; c; z), \\ F(a+,\, c-) &= F(a+1,\, b\,;\, c-1\,;\, z), \end{aligned}$$

and so on. Recall a contiguous function relation (see [4, p. 71]):

$$(2.1)\qquad (a-c+1)F = aF(a+) - (c-1)F(c-).$$

If we replace a, b, c, and z in (2.1) by a, $1-a$, b, and $\frac{1}{2}$ respectively, and apply (1.3) to the resulting equation, with the aid of Legendre duplication formula for the Gamma function:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.2}$$

we obtain

(2.3)

$$\begin{aligned}
&{}_2F_1\left(1+a,\, 1-a\,;\, b\,;\, \frac{1}{2}\right) \\
&= \frac{a-b+1}{a}\,{}_2F_1\left(a,\, 1-a\,;\, b\,;\, \frac{1}{2}\right) + \frac{b-1}{a}\,{}_2F_1\left(a,\, 1-a\,;\, b-1\,;\, \frac{1}{2}\right) \\
&= \frac{a-b+1}{a\cdot 2^{b-1}}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(\frac{b+a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)} + \frac{1}{a\cdot 2^{b-2}}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(\frac{b+a-1}{2}\right)\Gamma\left(\frac{b-a}{2}\right)}.
\end{aligned}$$

Setting $b = a+2$ and $b = a+3$ in (2.3) and recalling the generalized binomial coefficient

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{(-1)^n(-\alpha)_n}{n!}, \tag{2.4}$$

we readily obtain

$$\sum_{k=0}^{\infty}(-1)^k\binom{a-1}{k}\frac{1}{2^k(a+k+1)} = \frac{2^a\Gamma(a)\Gamma(a+1)}{\Gamma(2a+1)} - \frac{1}{a\cdot 2^a}, \tag{2.5}$$

which was also deduced by Srivastava [5];

$$\sum_{k=0}^{\infty}(-1)^k\binom{a}{k}\frac{k}{2^k(a+k)(a+k+1)} = \frac{2^{a-1}\{\Gamma(a+1)\}^2}{\Gamma(2a+2)} - \frac{1}{2^{a+1}}. \tag{2.6}$$

Lavoie *et al.* [3] obtained the following summation formula contiguous to (1.3):

(2.7)

$$\begin{aligned}
&{}_2F_1\left(a,\, 2-a\,;\, b\,;\, \frac{1}{2}\right) \quad (b \neq 0,\, -1,\, -2,\, \ldots) \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{2^{b-2}(1-a)}\left\{\frac{1}{\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b+a-1}{2}\right)} - \frac{1}{\Gamma\left(\frac{b-a+1}{2}\right)\Gamma\left(\frac{b+a-2}{2}\right)}\right\}.
\end{aligned}$$

Recall a contiguous function relation (see Cho *et al.* [1]):

$$(2.8)\quad (C-A-1)F=(C-A-B-1)F(A+)+B(1-z)F(A+,\,B+),$$

which, for the case $A=a$, $B=2-a$, $C=b$, and $z=\frac{1}{2}$, yields

$$(2.9)\quad \begin{aligned}{}_2F_1\left(1+a,\,3-a\,;\,b\,;\,\frac{1}{2}\right)&=\frac{2(a-b+1)}{a-2}\,{}_2F_1\left(a,\,2-a\,;\,b\,;\,\frac{1}{2}\right)\\&\quad+\frac{2(b-3)}{a-2}\,{}_2F_1\left(1+a,\,2-a\,;\,b\,;\,\frac{1}{2}\right).\end{aligned}$$

Recalling another contiguous function relation (see [4, p. 71]):

$$(2.10)\qquad (C-A-B)F=(C-A)F(A-)-B(1-z)F(B+),$$

and setting $A=1+a$, $B=1-a$, $C=b$, and $z=\frac{1}{2}$, and applying (1.3) and (2.3) to the resulting equation, we obtain

(2.11)

$$\begin{aligned}{}_2F_1\left(1+a,\,2-a\,;\,b\,;\,\frac{1}{2}\right)&=\frac{2(a-b+1)(a+b-2)}{a(a-1)}\,\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b+a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\\&\quad+\frac{4(b-2)}{a(a-1)}\cdot\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b+a-1}{2}\right)}.\end{aligned}$$

Finally setting (2.7) and (2.11) in (2.9), we find that

(2.12)

$$\begin{aligned}&{}_2F_1\left(1+a,\,3-a\,;\,b\,;\,\frac{1}{2}\right)\\&=\frac{2(a-b+1)(a+b-2)(a+2b-6)}{a(a-1)(a-2)}\,\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b+a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\\&\quad+\frac{4}{(a-1)(a-2)}\left\{\frac{2(b-2)(b-3)}{a}-a+b-1\right\}\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b+a-1}{2}\right)},\end{aligned}$$

which, for $b=a+3$, yields

$$(2.13)\quad \begin{aligned}&{}_2F_1\left(1+a,\,3-a\,;\,a+3\,;\,\frac{1}{2}\right)\\&=\frac{(a+1)(a+2)}{(a-1)(a-2)}\left[\frac{a+2}{2^{a-2}}-\frac{3\cdot 2^{a+1}\cdot\{\Gamma(a+1)\}^2}{\Gamma(2a+1)}\right],\end{aligned}$$

which, in view of (2.4), can be written in the following equivalent form:

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k\binom{a-2}{k}\frac{k}{2^k(a+k)(a+1+k)}\\ =\frac{3\cdot 2^a\cdot\{\Gamma(a+1)\}^2}{(a-1)\,\Gamma(2a+1)}-\frac{a+2}{(a-1)\,2^{a-1}}.\end{aligned} \tag{2.14}$$

Setting $a = n \in \mathbf{N}$ in (2.5), (2.6), and (2.14), and considering the following facts:

$$\binom{n}{k}=0 \quad (k>n) \tag{2.15}$$

and

$$\Gamma(1)=1, \quad \Gamma(n+1)=n! \;\; (n\in\mathbf{N}\cup\{0\}), \tag{2.16}$$

we immediately reach at the identities (1.1), (1.11) and (1.12).

We conclude this paper by noting that Lavoie *et al.* [3] have evaluated the following ${}_2F_1(1/2)$'s expressed in terms of Gamma function as some limiting cases of their main results:

$${}_2F_1\left(a,\, i-a\,;\, b\,;\, \frac{1}{2}\right) \quad (i=0,\, \pm1,\, \pm2,\, \pm3,\, \pm4,\, \pm5), \tag{2.17}$$

which, the cases $i = 3$ and $i = 4$ with replaced a by $a + 1$, also yields (2.11) and (2.12).

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