Comm. Korean Math. Soc. 14 (1999), No. 3, pp. 635–639

AN IDENTITY INVOLVING THE GENERALIZED HYPERGEOMETRIC SERIES

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Abstract. We provide a generalization of Whipple's quadratic transformation formula for ${}_3F_2$.

The generalized hypergeometric function with p numerator and q denominator parameters is defined by

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix};z\right] &= {}_pF_q\left[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z\right]\\
&= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},
\end{aligned}
$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial) is defined by, α any complex number,

$$
(1)\qquad (\alpha)_n := \begin{cases}\alpha(\alpha+1)\dots(\alpha+n-1), & \text{if } n=1,2,3,\dots,\\ 1 & \text{if } n=0.\end{cases}
$$

Using the fundamental property $\Gamma(z+1)=z\Gamma(z)$ of the Gamma function Γ, $(\alpha)_n$ can be written in the form

$$
(2)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},
$$

Received December 7, 1998. Revised January 4, 1999.
1991 Mathematics Subject Classification: primary 33C20, secondary 33C10, 33C15.
Key words and phrases: hypergeometric series, Whipple's theorem, Saalschütz's theorem.

where Γ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$\{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}, \tag{3}$$

γ being the Euler-Mascheroni's constant defined by

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\,901\,532 \ldots . \tag{4}$$

From (1), we can easily deduce the following formula:

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \quad (0 \le k \le n). \tag{5}$$

Setting $\alpha = 1$ in (5) gives the natural property

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases} \tag{6}$$

In this note, a generalization of Whipple's quadratic transformation formula for ${}_3F_2$ is considered. Indeed, we obtain the following quadratic transformation formula:

$$\begin{aligned} &\sum_{k=0}^{[n/2]} {}_3F_2 \left[\begin{matrix} -k, \gamma - b + n - k, \gamma - c + n - k \\ n - 2k + 1, \gamma + n - k \end{matrix} ; 1 \right] \\ &\times \frac{(1-\gamma-n)_k \, (-x)^k}{k! \, (n-2k)! \, (1-\gamma+b-n)_k \, (1-\gamma+c-n)_k} \\ &= \frac{(1-x)^n}{n!} {}_3F_2 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}, 1-\gamma-n \\ 1-\gamma+b-n, 1-\gamma+c-n \end{matrix} ; \frac{-4x}{(1-x)^2} \right], \end{aligned} \tag{7}$$

where $[x]$ denotes the greatest integer less than or equal to x and n is a nonnegative integer.

For convenience, let $I_n(\gamma, b, c; x)$ be the right member of (7). We find that, by using the following identity

$$(\alpha)_{2k} = 2^{2k}\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha}{2}+\frac{1}{2}\right)_k, \tag{8}$$

$$I_n(\gamma, b, c; x) = \frac{1}{n!}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}\,(1-\gamma-n)_k\,(-1)^k\,x^k\,(1-x)^{n-2k}}{k!\,(1-\gamma+b-n)_k\,(1-\gamma+c-n)_k}.$$

Recalling the generalized binomial theorem

$$(1-z)^{-a} = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k \quad (|z|<1), \tag{9}$$

we have

$$(1-x)^{n-2k} = \sum_{j=0}^{n-2k}\frac{(2k-n)_j}{j!}x^j.$$

We can then write $I_n(\gamma, b, c; x)$ as a double series as follows:

$$I_n(\gamma, b, c; x) = \frac{1}{n!}\sum_{k=0}^{[n/2]}\sum_{j=0}^{n-2k}\frac{(-n)_{2k}\,(1-\gamma-n)_k\,(-1)^k\,(2k-n)_j\,x^{k+j}}{k!\,j!\,(1-\gamma+b-n)_k\,(1-\gamma+c-n)_k},$$

which, upon applying one of the formal manipulations for double series:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}A_{j,k} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}A_{j,k-j} \tag{10}$$

considering the following conventions:

$$(-n)_{2k} = 0 \text{ if } k \geq \frac{n+1}{2};\quad (2k-n)_j = 0 \text{ if } j \geq n-2k+1,$$

leads to

$$I_n(\gamma, b, c; x)$$
$$= \frac{1}{n!}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-n)_{2k-2j}\,(1-\gamma-n)_{k-j}\,(-1)^{k-j}\,(2k-2j-n)_j\,x^k}{(k-j)!\,j!\,(1-\gamma+b-n)_{k-j}\,(1-\gamma+c-n)_{k-j}}.$$

Now using (5) and (6), we obtain

$$I_n(\gamma,b,c;x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-k)_j\,(\gamma-b+n-k)_j(\gamma-c+n-k)_j}{(n-2k+1)_j\,(\gamma+n-k)_j\,j!}$$
$$\times\frac{(1-\gamma-n)_k\,(-x)^k}{k!\,(n-2k)!\,(1-\gamma+b-n)_k\,(1-\gamma+c-n)_k}$$
$$=\sum_{k=0}^{[n/2]}{}_3F_2\left[\begin{matrix}-k,\,\gamma-b+n-k,\,\gamma-c+n-k\\ n-2k+1,\,\gamma+n-k\end{matrix};1\right]$$
$$\times\frac{(1-\gamma-n)_k\,(-x)^k}{k!\,(n-2k)!\,(1-\gamma+b-n)_k\,(1-\gamma+c-n)_k},$$

which completes the proof of (7).

In order to consider a special case of (7), we first recall the Saalschütz's theorem (see Rainville [1, p. 87]): If k is a nonnegative integer and if A, B, C are independent of k, then

$$(11)\qquad {}_3F_2\left[\begin{matrix}-k,\,A,\,B\\ C,\,1-C+A+B-k\end{matrix};1\right]=\frac{(C-A)_k\,(C-B)_k}{(C)_k(C-A-B)_k}.$$

Note that the identity (11) holds even though A, B, C are dependent of k.

For an application of Saalschütz's theorem (11), let $A=c+n-k$, $B=b+n-k$, and $C=n-2k+1$, we find that
(A)

$$ {}_3F_2\left[\begin{matrix}-k,\,c+n-k,\,b+n-k\\ n-2k+1,\,b+c+n-k\end{matrix};1\right]=\frac{(1-c-k)_k\,(1-b-k)_k}{(n-2k+1)_k(1-b-c-n)_k}.$$

Setting $\gamma=b+c$ in the left member of (7) yields

$$(B)\qquad I_n(b+c,b,c;x)=\sum_{k=0}^{[n/2]}{}_3F_2\left[\begin{matrix}-k,\,c+n-k,\,b+n-k\\ n-2k+1,\,b+c+n-k\end{matrix};1\right]$$
$$\times\frac{(1-b-c-n)_k\,(-x)^k}{k!\,(n-2k)!\,(1-c-n)_k\,(1-b-n)_k}.$$

Setting (A) into (B) with the aid of the following identities:

$$(1-\alpha-k)_k = (-1)^k\,(\alpha)_k \quad \text{and} \quad (n-2k)!\,(n-2k+1)_k = (n-k)!,$$

we have

$$\begin{aligned} I_n(b+c,b,c;x) &= \sum_{k=0}^{n} \frac{(b)_k\,(c)_k\,(-1)^k\,x^k}{k!\,(n-k)!\,(1-c-n)_k\,(1-b-n)_k} \\ &= \frac{1}{n!}\,{}_3F_2\left[\begin{matrix} -n,\; b,\; c \\ 1-b-n,\; 1-c-n \end{matrix};\, x\right], \end{aligned}$$

with which, setting $\gamma = b+c$ in the right member of (7), we obtain a quadratic transformation identity for ${}_3F_2$:

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} -n,\; b,\; c \\ 1-b-n,\; 1-c-n \end{matrix};\, x\right] \\ &= (1-x)^n\,{}_3F_2\left[\begin{matrix} -\frac{1}{2}n,\; -\frac{1}{2}n+\frac{1}{2},\; 1-b-c-n \\ 1-b-n,\; 1-c-n \end{matrix};\, \frac{-4x}{(1-x)^2}\right], \end{aligned}$$

which is due to Whipple [2].

ACKNOWLEDGMENTS. The authors wish to acknowledge the financial support of the Korea Research Foundation made in the program year of 1998, Project No. 1998-015-D00022.

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