Comm. Korean Math. Soc. 14 (1999), No. 2, pp. 429–440

GLOBAL STABILITY OF SOLUTIONS OF AN ELECTROCHEMISTRY MODEL WITH A SINGLE REACTION

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ABSTRACT. In this paper an electrochemistry model which consists of three charged species is considered. A dissociation-association reaction is allowed to take place between these species. The species of ions diffuse owing to concentration gradients and migrate because of electric forces. We prove that any initial distribution of species concentrations will settle down to the unique steady state as time becomes large.

1. Introduction

We consider an electrochemistry model of three charged species which simultaneously undergo diffusion owing to concentration gradients, migration because of electric force and dissociation-association reaction (see [6]). Let u_1, u_2 and u_3 denote the density of the charged particles A, B and C, respectively, where A is a neutrally charged species, and B and C are some positively and negatively charged species, respectively. The electric potential is denoted by ϕ. We assume that the species undergo a dissociation-association reaction which is represented by

$$A^0 \rightleftharpoons p\,B^{z_2} + q\,C^{z_3}.$$

Here p and q are positive integers, and z_i is the charge of the i-th species for $i = 1, 2, 3$ such that $z_1 = 0$, $z_2 > 0$ and $z_3 < 0$. Since the charge is conserved, we have

$$p\,z_2 + q\,z_3 = 0. \tag{1.1}$$

Received August 31, 1996. Revised January 19, 1999.
1991 Mathematics Subject Classification: 35K60.
Key words and phrases: Lyapunov functional, Sobolev imbedding theorem.
This research was supported partially by GARC-KOSEF.

As for the reaction, we assume mass action kinetics. Hence the rate of reaction, r, is given by

$$r = k^f\, u_1 - k^r\, u_2^p\, u_3^q, \qquad k^f,\, k^r > 0 \tag{1.2}$$

where k^f and k^r are reaction rate constants for forward and reverse reactions, respectively. The conservation equations for all the species are then given by the following system of nonlinear partial differential equations

$$\begin{aligned}
u_{1,t} &= d_1\, \Delta u_1 - r, && (1.3)\\
u_{2,t} &= d_2\, \nabla \cdot (\nabla u_2 + z_2\, u_2\, \nabla \phi) + p\, r, && (1.4)\\
u_{3,t} &= d_3\, \nabla \cdot (\nabla u_3 + z_3\, u_3\, \nabla \phi) + q\, r, && (1.5)\\
\varepsilon\, \Delta \phi &= -\,(z_2\, u_2 + z_3\, u_3). && (1.6)
\end{aligned}$$

They are subject to the following boundary and initial conditions

$$\begin{aligned}
&\nabla u_1 \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,\ t > 0, && (1.7)\\
&(\nabla u_i + z_i\, u_i \nabla \phi) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,\ t > 0,\ i = 2,\, 3, && (1.8)\\
&\phi = 0 \quad \text{on } \Gamma_1, \quad \phi = \alpha \quad \text{on } \Gamma_2, \quad \frac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_3,\ t > 0, && (1.9)\\
&u_i(x, 0) = u_i^0(x) \geq 0, \quad i = 1,\, 2,\, 3 && (1.10)
\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector at $\partial\Omega$. Here we let Ω^0, ω_1, ω_2 be open, bounded, connected subsets of R^N, $N = 2$ such that $\overline{\omega_1},\ \overline{\omega_2} \subset \Omega^0$ and $\overline{\omega_1} \cap \overline{\omega_2} = \emptyset$. Let Γ_1, Γ_2 and Γ_3 be the smooth boundary of ω_1, ω_2 and Ω^0, respectively. Let $\Omega = \Omega^0 \setminus (\overline{\omega_1} \cup \overline{\omega_2})$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We shall study equation (1.1)-(1.10) in the domain $\Omega \times R^+$.

Without loss of generality we assume that α in (1.9) and ε in (1.6) are positive. In particular, the boundary condition (1.7)-(1.8) implies under the assumption that the flux of species given by $-\,d_i(\nabla u_i + z_i\, u_i \nabla \phi)$, where $d_i > 0$ is a diffusion constant, is zero on the boundary $\partial\Omega$ for $i = 1,\, 2,\, 3$.

The initial condition $\phi(x, 0) = \phi^0(x)$ is not prescribed because it can be obtained by solving the equation $\Delta \phi^0 = -\,(z_2\, u_2^0 + z_3\, u_3^0)$ which is subject to the boundary conditions $\phi^0|_{\Gamma_1} = 0,\ \phi^0|_{\Gamma_2} = \alpha,\ \partial \phi^0 / \partial \mathbf{n}|_{\Gamma_3} = 0$.

In addition, one can prove the local existence and uniqueness of solutions to the above initial-boundary value problem (1.1)-(1.10), provided that u_i^0 $(i = 1, 2, 3)$ are sufficiently smooth and satisfy the boundary conditions (1.7)-(1.8) at the boundary when $t = 0$. The solutions are smooth for $t > 0$ as long as they exist (see [4]). Furthermore, one can show that $u_i > 0$ for $0 < t < T$, where T is the maximal time of existence (see [1]).

In the stationary case, steady-state solutions which are obtained from the evolution problem (1.1)-(1.10) are equivalent to the solutions of the following nonlinear boundary value problem

$$d_1 \Delta \tilde{u}_1 - \tilde{r} = 0, \tag{1.11}$$

$$d_2 \nabla \cdot (\nabla \tilde{u}_2 + z_2 \tilde{u}_2 \nabla \tilde{\phi}) + p \tilde{r} = 0, \tag{1.12}$$

$$d_3 \nabla \cdot (\nabla \tilde{u}_3 + z_3 \tilde{u}_3 \nabla \tilde{\phi}) + q \tilde{r} = 0, \tag{1.13}$$

$$\varepsilon \Delta \tilde{\phi} = -(z_2 \tilde{u}_2 + z_3 \tilde{u}_3) \tag{1.14}$$

where $\tilde{r} = k^f \tilde{u}_1 - k^r \tilde{u}_2^p \tilde{u}_3^q$. The solutions are subject to the following boundary conditions

$$p \int_\Omega \tilde{u}_1 \, dx + \int_\Omega \tilde{u}_2 \, dx = C_2, \quad q \int_\Omega \tilde{u}_1 \, dx + \int_\Omega \tilde{u}_3 \, dx = C_3, \tag{1.15}$$

$$\nabla \tilde{u}_1 \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.16}$$

$$(\nabla \tilde{u}_i + z_i \tilde{u}_i \nabla \tilde{\phi}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \quad i = 2, 3, \tag{1.17}$$

$$\tilde{\phi} = 0 \quad \text{on} \quad \Gamma_1, \quad \tilde{\phi} = \alpha \quad \text{on} \quad \Gamma_2, \quad \frac{\partial \tilde{\phi}}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_3. \tag{1.18}$$

C_2 and C_3 in (1.15) are known positive constants which are determined by the initial ion concentrations. Equation (1.11)-(1.18) will be collectively referred to as the steady-state problem. In the equilibrium state we physically observe that the reaction term $\tilde{r}$ in equation (1.11)-(1.13) becomes zero. The fact was proved mathematically in the paper [1]:

LEMMA 1.1. *Let $\tilde{u}_i$ and $\tilde{\phi}$ be any solutions to the steady-state problem* (1.11)-(1.18). *Then $\tilde{r} = 0$.*

Moreover, the existence and uniqueness of steady-state solutions to the steady-state problem was proved in the paper [1]:

LEMMA 1.2. *Let C_2 and C_3 be given positive constants. Then the positive solution to the steady-state problem is unique.*

In this paper we shall prove the following global stability theorem:

THEOREM 1.1. *Let $p = q = 1$ in* (1.1). *Let u_i^0 $(i = 1, 2, 3)$ be bounded smooth functions defined on the domain $\Omega \subset R^2$ which satisfy the compatibility conditions $\nabla u_i^0 + z_i u_i^0 \nabla \phi^0 = 0$ on the boundary of Ω. The solutions to equation* (1.1)-(1.10) *then converge uniformly to the unique steady-state solutions to the steady-state problem* (1.11)-(1.18).

We now shall assume that Theorem 1.2 (see below) is valid, and use it to prove Theorem 1.1. The proof of Theorem 1.2 will be presented in the section 3.

THEOREM 1.2. *Let $p = q = 1$ in* (1.1). *Suppose that the hypotheses of* Theorem 1.1 *on u_i^0 $(i = 1, 2, 3)$ are satisfied. Smooth solutions to equation* (1.1)-(1.10) *then exist in $Q^* \equiv \bar{\Omega} \times [\delta, \infty]$, where $\delta > 0$ and*

$$\|u_i\|_{H^{j+\lambda,(j+\lambda)/2}(Q^*)} \leq M_{j,\lambda}$$

where $M_{j,\lambda} > 0$ depends on the initial data u_i^0, $0 < \lambda < 1$, and j is a positive integer.

We also introduce some definitions, which will be used in the section 3.

DEFINITIONS. Let $0 < \lambda < 1$ and $T > 0$ and, for $v(x, t)$ defined in $Q_T \equiv \Omega \times [0, T]$, we define the Hölder norms of v as follows:

$$\begin{aligned}
\|v\|_{C^\lambda, x} &\equiv \sup_{(x,t),(y,t)\in Q_T} \frac{|v(x, t) - v(y, t)|}{|x - y|^\lambda}, \\
\|v\|_{C^\lambda, t} &\equiv \sup_{(x,t),(x,\gamma)\in Q_T} \frac{|v(x, t) - v(x, \gamma)|}{|t - \gamma|^\lambda}, \\
\|v\|_{H^{\lambda,\lambda/2}(Q_T)} &\equiv \|v\|_{L^\infty(Q_T)} + \|v\|_{C^\lambda, x} + \|v\|_{C^{\lambda/2}, t}, \\
\|v\|_{H^{1+\lambda,(1+\lambda)/2}(Q_T)} &\equiv \|v\|_{L^\infty(Q_T)} + \|v\|_{C^{(1+\lambda)/2}, t} \\
&\quad + \sum_{|\alpha|=1} \|D^\alpha v\|_{H^{\lambda,\lambda/2}(Q_T)}.
\end{aligned}$$

In addition, we define the Sobolev norm of v as follows:

$$\|v\|_{W_p^{2,1}(Q_T)} \equiv \|v_t\|_{L^p(Q_T)} + \sum_{|\alpha|\leq 2} \|D^\alpha v\|_{L^p(Q_T)}.$$

We now shall cite the following two lemmas that are contained in [3, pp. 12-14], which allow us to estimate Hölder norm bounds on u_i and ϕ of equation (1.1)-(1.10).

LEMMA 1.3. *Suppose that the hypotheses of* Lemma 3.1 *in the section 3 are satisfied. Then for any $p > 1$, there exists a constant $M_p > 0$ which may depend on p and the initial data u_i^0 $(i = 1, 2, 3)$, but which may be independent of t for $t \in (0, T)$ such that*

$$\|\phi_t\|_{L^p(\Omega)} \leq M_p.$$

LEMMA 1.4. *Suppose that the hypotheses of* Lemma 3.1 *in the section 3 are satisfied. Then for any $0 < \lambda < 1$, there exists a constant $k_\lambda^* > 0$ which is independent of t for $t \in (0, T)$ such that*

$$\|\phi\|_{H^{1+\lambda,(1+\lambda)/2}(\bar{\Omega}\times[0,T))} \leq k_\lambda^*.$$

2. Lyapunov functional and the proof of Theorem 1.1

In the case of a damped oscillator without external forcing, energy decreases with time. Such monotone behavior makes the dynamics of the system relatively simple. The dynamics of the solution to any system of equations will be simpler to understand if we can identify a scalar functional that has a monotone behavior as time increases. Such a functional is known as Lyapunov functional.

As for our equation (1.1)-(1.10), such a Lyapunov functional exists and is given by

$$V(t) = \int_\Omega \left[\frac{\varepsilon}{2}|\nabla\phi|^2 + \sum_{i=1}^{3} u_i \log(\alpha_i u_i) - \sum_{i=1}^{3} u_i - \varepsilon \operatorname{div}(\phi\nabla\phi)\right] dx$$

where $\alpha_1 = k^f$, $\alpha_2 = k^r$ and $\alpha_3 = k^r$. Observe that V is well defined for $t > 0$, since $u_i > 0$. Let $\mathbf{F}_i \equiv d_i(\nabla u_i + z_i u_i \nabla\phi)$, which is the negative of the flux of species i.

By differentiating V with respect to t, integrating by parts and using the boundary conditions for ϕ, after some manipulations we obtain

$$\dot{V}(t) = -\int_\Omega \sum_{i=1}^{3} \frac{1}{d_i\, u_i} |\mathbf{F}_i|^2 \, dx - \int_\Omega r \, \left[\log(k^f u_1) - \log(k^r u_2 u_3)\right] \, dx.$$

With

$$\int_\Omega r \, \left[\log(k^f u_1) - \log(k^r u_2 u_3)\right] \, dx > 0,$$

the above estimate yields

$$\dot{V}(t) \le -\int_\Omega \sum_{i=1}^{3} \frac{1}{d_i\, u_i} |\mathbf{F}_i|^2 \, dx \le 0. \tag{2.1}$$

Consequently, $V(t)$ is a Lyapunov functional for equation (1.1)-(1.10).

Furthermore, we show that V is bounded below, which leads to the following lemma:

LEMMA 2.1.

$$\lim_{t\to\infty} V(t) = l_1.$$

PROOF. Guided by the ideas contained in [3, pp. 3-4], we shall prove the lemma. In exactly the same way as in [3, p. 3], one can show that

$$\varepsilon \int_\Omega \nabla \cdot (\phi \nabla \phi) \, dx \le k_1 + \frac{\varepsilon}{4} \|\nabla \phi\|_{L^2}^2$$

where k_1 is a positive constant.

We observe from (1.3)-(1.5) that $p \int_\Omega u_1(x,t)\, dx + \int_\Omega u_2(x,t)\, dx = C_2$ and $q \int_\Omega u_1(x,t)\, dx + \int_\Omega u_3(x,t)\, dx = C_3$, where C_2 and C_3 are positive constants which depend only on the initial conditions. Hence

$$\int_\Omega u_i(x,t)\, dx \le \hat{C} \equiv \max\{C_2, C_3\}, \quad \mathrm{i} = 1,\, 2,\, 3. \tag{2.2}$$

Using equation (2.1)-(2.2) and the two facts: $V(t) \le V(0)$ and the function $u \log u$ is bounded below for $u \in [0, \infty)$, we conclude that

$$\int_\Omega |\nabla\phi(x,t)|^2\,dx \le M_1, \tag{2.3}$$

$$\int_\Omega u_i(x,t)\,\log(\alpha_i\,u_i(x,t))\,dx \le M_1, \tag{2.4}$$

where M_1 is a positive constant independent of t as long as u_i and ϕ exist. From (2.1), (2.3) and (2.4), $V(t)$ is bounded below. Since V is a decreasing function in t, $\lim_{t\to\infty} V(t) = l_1$ exists. □

LEMMA 2.2. *$\mathbf{F}_i$ converges to zero uniformly on $\bar\Omega$ as $t \to \infty$ for $i = 1, 2, 3$.*

PROOF. By Theorem 1.2 we define $\|u_i\|_{L^\infty} \le M^*$ and $d^* \equiv \max\{d_i\}$ for all i. From (2.1), let $W(t)$ be defined for $t \ge 1$ by $\dot W(t) = -\int_\Omega \sum_{i=1}^3 \frac{1}{d^* M^*}$ $|\mathbf{F}_i|^2\,dx$ and $W(1) = V(1)$. $W(t)$ is then nonincreasing in t and $V(t) \le W(t)$ for $t \ge 1$, from which it follows that $\lim_{t\to\infty} W(t) = l_2$ exists.

Furthermore it follows from Theorem 1.2 that $|\ddot W(t)|$ is bounded for $t \ge 1$. Hence we can conclude that $\lim_{t\to\infty} \dot W(t) = 0$, which yields

$$\lim_{t\to\infty}\int_\Omega |\mathbf{F}_i|^2\,dx = 0 \quad \text{for all} \quad i. \tag{2.5}$$

By Theorem 1.2, the Arzela-Ascoli theorem and (2.5), we have $\lim_{t\to\infty} \mathbf{F}_i = 0$ in L^∞ norm. □

Now we are ready to prove the main theorem.

PROOF OF THEOREM 1.1. Let $\{t_k\}$ be a sequence with $\lim_{k\to\infty} t_k = \infty$. There exists a subsequence from Theorem 1.2, which is also denoted by $\{t_k\}$, such that

$$\lim_{k\to\infty} u_i(x,t_k) = U_i(x) \quad \text{in} \quad C^2(\bar\Omega) \quad \text{norm},$$
$$\lim_{k\to\infty} \phi(x,t_k) = \Phi(x) \quad \text{in} \quad C^2(\bar\Omega) \quad \text{norm}.$$

From Lemma 2.2 and the definition of $\mathbf{F}_i$, U_i and Φ satisfy the following equations

$$d_i \nabla \cdot (\nabla U_i + z_i U_i \nabla \Phi) = 0 \quad \text{for all} \quad i,$$
$$\varepsilon \Delta \Phi = -(z_2 U_2 + z_3 U_3),$$

which show along with Lemma 1.1 that U_i and Φ are steady-state solutions to the steady-state problem (1.11)-(1.18). It follows from Lemma 1.2 that $\tilde{u}_i = U_i$ and $\tilde{\phi} = \Phi$, for $i = 1, 2, 3$. In summary, we have shown that for every sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, there exists a subsequence, denoted by $\{t_k\}$, such that $u_i(x, t_k)$ and $\phi(x, t_k)$ converge to the unique steady-state solutions $\tilde{u}_i$ and $\tilde{\phi}$ as $k \to \infty$. □

3. Proof of the Theorem 1.2

In this section we shall outline the proof of Theorem 1.2. It should be pointed out that an idea of proving which is similar to the one in our proof below was employed by Choi and Lui (see [2]-[3]) to obtain $L^p(\Omega)$-norm and Hölder norm bounds on u_i and ϕ. However, Choi and Lui's proof required the assumption (1.1) on the coefficients of differential equations, which is not used in this section.

3.1. L^p-estimates for u_i

To prove that the $L^p(\Omega)$-norm of the solutions u_i $(i = 1, 2, 3)$ is bounded independently of t for $1 \leq p \leq \infty$ when $N = 2$, one can employ an argument similar to the one given in [3, pp. 7-12]. For example, one can first show that the $L^2(\Omega)$-norm of u_i is bounded independently of t. Second, using $L^2(\Omega)$-norm bounds, one can prove time-independent bounds of $L^4(\Omega)$-norm of u_i, which Choi and Lui' proof didn't require. One can then bootstrap the results to obtain higher L^p-norm bounds. Since the proof of Lemma 3.1 (see below) is quite technical and routine, we skip it.

LEMMA 3.1. *Let $p = q = 1$, $N = 2$ and $T > 0$ be the maximal time of existence of solutions to equation (1.1)-(1.10). Then there exists a constant $M > 0$, which may depend on the initial data u_i^0, but which may be independent of t for $t \in (0, T)$, such that $\|u_i\|_{L^\infty}$ is bounded above by M for $i = 1, 2, 3$.*

3.2. Hölder norm bound for u_i

Before proceeding with our derivation of time-independent a priori bounds in Hölder norms for solutions to equation (1.1)-(1.10), we first let $v_i \equiv u_i\, e^{z_i\,\phi}$ for $i = 2,\, 3$, which is guided by the ideas contained in [3]. Then v_i satisfies the equation

$$\begin{aligned}\frac{\partial v_i}{\partial t} &= d_i\,\Delta v_i - d_i\, z_i\,\nabla\phi\cdot\nabla v_i + z_i\,\phi_t\, v_i + (k^f\, u_1 - k^r\, u_2\, u_3)\, e^{z_i\,\phi},\\ \frac{\partial v_i}{\partial \mathbf{n}} &= 0 \quad \text{on} \quad \partial\Omega, \quad t > 0.\end{aligned}$$

From now on we shall work with v_i rather than $u_i\,(i = 2,\, 3)$ because it satisfies the zero-Neumann boundary condition.

LEMMA 3.2. *Suppose that the hypotheses of* Lemma 3.1 *are satisfied.* $\|\nabla v_i\|_{L^2(\Omega)}$ *is bounded above by* M*, where* $M > 0$ *may depend on the initial data* $u_i^0\,(i = 2,\, 3)$*, but it may be independent of* t *for* $t \in (0,\, T)$.

PROOF.

$$\begin{aligned}&\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}\,|\nabla v_i|^2\, dx\\ &\quad= \int_\Omega \left[-\, d_i\,(\Delta v_i)^2 + d_i\, z_i\,\nabla\phi\cdot\nabla v_i\,\Delta v_i - z_i\,\phi_t\, v_i\,\Delta v_i\right]\, dx\\ &\qquad - \int_\Omega \Delta v_i\,\left[(k^f\, u_1 - k^r\, u_2\, u_3)\, e^{z_i\phi}\right]\, dx.\end{aligned}$$

In exactly the same way as in [3, p. 14], one can prove that

$$\begin{aligned}\int_\Omega &\left[-\, d_i(\Delta v_i)^2 + d_i\, z_i\,\nabla\phi\cdot\nabla v_i\Delta v_i - z_i\,\phi_t\, v_i\,\Delta\right]\, dx\\ &\qquad\qquad\le -\, k_1\,\|\nabla v_i\|_2^2 + k_2.\end{aligned}$$

Lemmas 3.1 and 1.3 allow us to estimate the following inequality:

$$\left| -\int_\Omega \Delta v_i \left[(k^f u_1 - k^r u_2 u_3) e^{z_i \phi} \right] dx \right| \leq k_3 \|\Delta v_i\|_2.$$

Thus the result follows. □

Now we are ready to bootstrap the $L^p(\Omega)$-norm bound on u_i to obtain a Hölder norm bound on u_i and ϕ.

PROOF OF THE THEOREM 1.2. The first step is to obtain estimates for u_i, for $i = 2, 3$. The proof for u_1 is similar and simpler. Finally we combine these estimates to prove Theorem 1.2.

Consider the sequence of overlapping cylinders $Q_N \equiv \Omega \times [N, N+2]$, where $N = 0, 1, 2, \cdots$. In Q_N, let a_i^N and b_i^N satisfy the partial differential equations (3.1)-(3.3) and (3.4)-(3.6), respectively:

$$\begin{aligned}
&(3.1) \qquad & \partial a_i^N/\partial t &= d_i \Delta a_i^N(x, t) + \Gamma(x, t+N), \\
&(3.2) \qquad & \partial a_i^N/\partial \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\
&(3.3) \qquad & a_i^N(x, 0) &= 0, \\
&(3.4) \qquad & \partial b_i^N/\partial t &= d_i \Delta b_i^N, \\
&(3.5) \qquad & \partial b_i^N/\partial \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\
&(3.6) \qquad & b_i^N(x, 0) &= v_i(x, N)
\end{aligned}$$

where

$$\Gamma(x, t+N) \equiv \left[-d_i z_i \nabla\phi \cdot \nabla v_i + z_i \phi_t v_i + (k^f u_1 - k^r u_2 u_3) e^{z_i\phi} \right] \Big|_{(x,t)}.$$

We then decompose

$$v_i(x, t+N) = a_i^N(x, t) + b_i^N(x,t) \quad \text{for } 0 < t < 2.$$

Let $Q_N^* \equiv \Omega \times [N+1/2, N+2]$. From (3.4)-(3.6),

$$\|b_i^N\|_{H^{j+\lambda, (j+\lambda)/2}(\bar{Q}_0^*)} \leq k_{j,\lambda} \tag{3.7}$$

where j is any positive integer, $0 < \lambda < 1$, and $k_{j,\lambda} > 0$ is independent of N, because $\|v_i(\cdot, N)\|_\infty$ is bounded independently of N.

We now estimate a_i^N and its higher derivatives. In the following argument, k_i, $i = 1, 2, \cdots$ will denote positive constants independent of N. Let us consider the nonhomogeneous term $\Gamma(x, t+N)$ in (3.1). By Lemmas 1.3 and 3.2, it can be checked that $\|\Gamma(x, t+N)\|_{L^2(Q_0)}$ is bounded independently of N. Therefore, using Agmon-Douglas-Nirenberg type estimates and Sobolev imbedding theorem (see [5, p. 80]), we see that equation (3.1)-(3.3) has a unique solution $a_i^N \in W_2^{2,1}(Q_0)$ and $\|\nabla a_i^N\|_{L^4(Q_0)} \le k_1$. Combining this with (3.7) yields $\|\nabla v_i\|_{L^4(Q_N^*)} \le k_2$. Repeating the same argument with these improved estimates on v_i, one can check that problem (3.1)-(3.3) has a unique solution $a_i^N \in W_p^{2,1}(Q_0)$ and $\|a_i^N\|_{W_p^{2,1}(Q_0)} \le k_p$ for any $p > 1$. Thus, for any $\lambda \in (0, 1)$, by increasing p if necessary (see [5, p. 80]) the Sobolev imbedding theorem implies that there exists a constant k_λ such that

$$\|a_i^N\|_{H^{1+\lambda,(1+\lambda)/2}(\bar{Q}_0^*)} \le k_\lambda \tag{3.8}$$

where $k_\lambda > 0$ is independent of N.

From (3.7)-(3.8) we have $\|v_i\|_{H^{1+\lambda,(1+\lambda)/2}(\bar{Q}_N^*)} \le M_\lambda$, where $M_\lambda > 0$ is independent of N. Since $u_i = v_i\, e^{-z_i\phi}$, Lemma 1.4 implies that the above estimate on v_i is also valid with v_i replaced by u_i and a different M_λ. Because Q_N^*, $N = 1, 2, \cdots$ overlap one another, there exists a constant $M_{1,\lambda} > 0$, which is independent of N and satisfies

$$\|u_i\|_{H^{1+\lambda,(1+\lambda)/2}(\bar{\Omega}\times[\delta,T))} \le M_{1,\lambda}.$$

Using the same argument, one can show that $\|u_1\|_{H^{1+\lambda,(1+\lambda)/2}(\bar{Q}_N^*)}$ is bounded. Repeating the above bootstrap argument with new estimates on u_i ($i = 1, 2, 3$), we can obtain a priori bounds on $\|\phi\|_{H^{k+\lambda,(k+\lambda)/2}(\bar{\Omega}\times[\delta,\infty))}$ and $\|u_i\|_{H^{k+\lambda,(k+\lambda)/2}(\bar{\Omega}\times[\delta,\infty))}$ to any desired order k. The above estimates imply that solutions u_i and ϕ exist for all time so that $T = \infty$. $\square$

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