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THE EXISTENCE OF SOLUTION OF SEMILINEAR ELLIPTIC EQUATION

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ABSTRACT. This paper is concerned with the existence of positive solution of a semilinear elliptic equation with homogeneous Dirichlet boundary condition.

1. Introduction

This paper is concerned with the existence of positive solution of semilinear elliptic equation with homogeneous Dirichlet boundary condition

$$(P)\qquad \begin{cases} -\triangle u = f(x,u) & \text{in } \Omega \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary and the function $f(x,u)$: $\Omega \times [0,\infty) \to R$ satisfies the following assumption:

(H1) For almost all $x \in \Omega$, the function $u \longmapsto f(x,u)$ is continuous on $[0,\infty)$ and for each $\delta > 0$, there is a constant $C_\delta \geq 0$ such that

$$f(x,u) \geq -C_\delta u \text{ for almost all } x \in \Omega, \text{ for all } u \in [0,\delta].$$

(H2) For each $u \geq 0$, the function $x \longmapsto f(x,u)$ belongs to $L^\infty(\Omega)$.

We introduce the measurable function

$$a_0(x) = \lim_{u \downarrow 0} \frac{f(x,u)}{u}$$

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$$a_\infty(x) = \lim_{u\uparrow\infty} \frac{f(x,u)}{u}$$

so that $-\infty < a_0(x) \leq +\infty$, *and* $-\infty \leq a_\infty(x) < +\infty$.

Previously, Brezis and Oswald [3] treated the case of sublinear elliptic equation. More precisely, in [3] a solution of (P) was shown to exist at most one solution by assuming (H1), (H2). Moreover, if a solution of (P) exists and the function $u \longmapsto f(x,u)/u$ is decreasing on $(0,\infty)$, then the facts that

(1) $$\lambda_1(-\triangle - a_0(x)) < 0$$

(2) $$\lambda_1(-\triangle - a_\infty(x)) > 0$$

are known. By the strong maximum principle,we know that u is a positive solution on Ω. Here $\lambda_1(-\triangle - a(x))$ denotes the first eigenvalue of $-\triangle - a(x)$ with zero Dirichlet condition on $\partial\Omega$. In the present paper, we shall generalize H. Brezis and L. Oswald's result [3] under the critical growth condition on $f(x,u)$. In [3], the condition(H3) is assumed only for p=2.

(H3) There is a constant $C > 0$ such that

$$f(x,u) \leq C(u^{p-1} + 1)$$

for almost all $x \in \Omega$, for all $u \geq 0$ where $2 \leq p < \frac{2N}{N-2}$ if $N \geq 3$ and $2 \leq p < \infty$ if $N = 2$.

Instead of $a_0(x)$ above, we replace it by

(3) $$a_0(x) = \liminf_{u\downarrow 0} \frac{f(x,u)}{u}.$$

Let us assume that

$$\limsup_{u\uparrow\infty} \frac{2F(x,u)}{u^2} \leq \lambda_1(-\triangle)$$

uniformly for almost all $x \in \Omega$ where $F(x,u) = \int_0^u f(x,t)dt$ and

$$\limsup_{u \uparrow \infty} \frac{2F(x,u)}{u^2} < \lambda_1(-\triangle) \tag{4}$$

on a subset of Ω of positive measure (abbr., $\lambda_1(-\triangle) = \lambda_1$).

From (H1), there is a constant C such that $a_0(x) \geq -C$ and from (4), we may assume that there is a function $\alpha(x) \in L^\infty(\Omega)$ such that

$$\limsup_{u \uparrow \infty} \frac{2F(x,u)}{u^2} \leq \alpha(x) \leq \lambda_1 \tag{5}$$

uniformly for almost all $x \in \Omega$.

For example, if $f(x,u) = e^{-|x|u^2} u^{p-1}$, since

$$\begin{aligned} F(x,u) &= \int_0^u e^{-|x|t^2} t^{p-1} dt \\ &= \frac{1}{2} \int_0^{u^2} e^{-|x|s} s^{\frac{p}{2}-1} ds \\ &= \frac{-1}{2|x|} \frac{u^{p-2}}{e^{|x|u^2}} + O\left(\frac{u^{p-4}}{|x|^2 e^{|x|u^2}} \right), \end{aligned}$$

then

$$\lim_{u \uparrow \infty} \frac{2F(x,u)}{u^2} = 0 \leq \lambda_1(-\triangle).$$

Thus we can take the functions of this type $f(x,u) = e^{-|x|u^2} u^{p-1}$ satisfying both conditions (H_3) and (4). This function also satisfies both conditions (H_1) and (H_2).

Our main result is the following:

MAIN THEOREM. *Under the conditions (H_1), (H_2), (H_3), (1) and (4), there exists a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (P).*

We will prove the theorem using following Lemma.

LEMMA. *Assume that the conditions* (4) *and* (5) *hold. Then there is* $\delta > 0$ *such that, for every* $u \in H_0^1(\Omega)$,

$$\psi(u) = \frac{1}{2}\int_\Omega [|\nabla u|^2 - \alpha(x)u^2] \geq \delta \int_\Omega |\nabla u|^2.$$

PROOF. It follows from Poincaré's inequality that

$$\psi(u) \geq \frac{1}{2}\int_\Omega [|\nabla u|^2 - \lambda_1 u^2] \geq 0.$$

If $\psi(u) = 0$, then $\int |\nabla u|^2 = \int \lambda_1 u^2$ and thus

$$0 = \psi(u) = \frac{1}{2}\int_\Omega (\lambda_1 - \alpha(x))u^2.$$

Since $\lambda_1 > \alpha(x)$ on a subset of Ω of positive measure, $u = 0$ on a subset of Ω of positive measure. By the unique continuation property, we obtain $u \equiv 0$. Assume now that the conclusion is false. Then there is a sequence (u_n) in $H_0^1(\Omega)$ such that

$$\int_\Omega |\nabla u_n|^2 = 1, \ u_n \rightharpoonup u \ \text{ in } H_0^1(\Omega), \ u_n \to u \ \text{ in } \ L^2(\Omega)$$

and $0 \leq \psi(u_n) \to 0$ as $n \to \infty$. We obtain

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 \geq \int_\Omega |\nabla u|^2$$

and

$$\int_\Omega |\nabla u_n|^2 \to \int_\Omega \alpha(x)u^2.$$

Hence $0 \leq \psi(u) \leq 0$, i.e., $\psi(u) = 0$. Thus $u \equiv 0$. But $1 = \int_\Omega |\nabla u_n|^2 \to 0$ which is impossible. □

2. Proof of main theorem

We consider the functional $E : H_0^1(\Omega) \to R \cup \{\infty\}$ defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) \text{ for all } u \in H_0^1(\Omega),$$

where $F(x,u) = \int_0^u f(x,t)dt$ and $f(x,u)$ is extended to be $f(x,0)$ for $u \leq 0$. Note that $E(u)$ is well-defined, since $F(x,u) \leq C(\frac{1}{p}|u|^p + |u|)$ for all $x \in \Omega$, for all $u \in R$. We claim that the following:
(8) E is coercive on $H_0^1(\Omega)$,
(9) E is lower semicontinuous for the weak $H_0^1(\Omega)$ topology,
(10) There is some $\phi \in H_0^1(\Omega)$ such that $E(\phi) < 0$.

Proof of (8). From the conditions ($H3$) and (5), there is a function $\beta(x) \in L^1(\Omega)$ such that

$$F(x,u) \leq (\alpha(x) + \lambda_1\delta)\frac{u^2}{2} + \beta(x).$$

Thus

$$\begin{aligned}
E(u) &= \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - F(x,u)\right] \\
&\geq \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \alpha(x)\frac{u^2}{2} - \frac{\lambda_1\delta}{2}u^2 - \beta(x)\right] \\
&= \psi(u) - \int_\Omega \frac{\lambda_1\delta}{2}u^2 - \int_\Omega \beta(x) \\
&\geq \delta\int_\Omega |\nabla u|^2 - \frac{\lambda_1\delta}{2}\int_\Omega u^2 - \int_\Omega \beta(x) \\
&= \delta\int_\Omega \left[|\nabla u|^2 - \frac{\lambda_1}{2}u^2\right] - \int_\Omega \beta(x) \\
&\geq \delta\int_\Omega \left[|\nabla u|^2 - \frac{|\nabla u|^2}{2}\right] - \int_\Omega \beta(x) \\
&= \frac{\delta}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \beta(x).
\end{aligned}$$

Thus $E(u)$ is coercive on $H_0^1(\Omega)$ under the norm $\|u\|_{H_0^1} = [\int_\Omega |\nabla u|^2]^{1/2}$.□

PROOF OF (9). Let $u_n \rightharpoonup u$ converge weakly to u in $H_0^1(\Omega)$. By Sobolev's imbedding theorem, passing to a subsequence if necessary, we may suppose that $u_n \to u$ in $L^p(\Omega)$, $u_n \to u(x)$ for almost all $x \in \Omega$ and $|u_n(x)| \leq h(x)$ for some $h \in L^p(\Omega)$. Then it follows from (H3) that

$$|F(x, u_n(x))| \leq C(h(x)^p + h(x)).$$

Since the right side of the above inequality is in $L^1(\Omega)$, we have, by Lebesque dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega F(x, u_n) = \int_\Omega F(x, u).$$

Thus

$$\begin{aligned}\lim_{n\to\infty} \inf E(u_n) &= \lim_{n\to\infty} \inf\left(\int_\Omega \frac{1}{2}|\nabla u_n|^2 - F(x, u_n)\right)\\ &\geq \lim_{n\to\infty} \inf \int_\Omega \frac{1}{2}|\nabla u_n|^2 - \lim_{n\to\infty} \int_\Omega F(x, u_n)\\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x, u) = E(u).\end{aligned}$$

□

PROOF OF (10). We fix any $\phi \in H_0^1(\Omega)$ satisfying

$$\int_\Omega |\nabla \phi|^2 - \int_{[\phi\neq 0]} a_0 \phi^2 < 0.$$

Such ϕ always exists by assumption (1). We may always assume that $\phi > 0$ and that $\phi \in L^\infty(\Omega)$. Otherwise, we replace ϕ by $|\phi|$ and truncate ϕ. We note that

$$\lim_{u\downarrow 0} \inf \frac{F(x, u)}{u^2} \geq \frac{1}{2} a_0(x)$$

and thus

$$\lim_{\varepsilon\downarrow 0} \inf \frac{F(x, \varepsilon\phi)}{\varepsilon^2} \geq \frac{1}{2} a_0(x)\phi^2(x) \quad \text{for almost all} \quad x \in [\phi \neq 0].$$

On the other hand, we deduce from the condition (H1) that

$$\frac{F(x,\varepsilon\phi)}{\varepsilon^2} \geq -C\phi^2 \geq -C.$$

Therefore, by Fatou's lemma, it follows

$$\liminf_{\varepsilon\downarrow 0} \int_{[\phi\neq 0]} \frac{F(x,\varepsilon\phi)}{\varepsilon^2} \geq \frac{1}{2}\int_{[\phi\neq 0]} a_0\phi^2,$$

from which

$$\liminf_{\varepsilon\downarrow 0} \int_{\Omega} \frac{F(x,\varepsilon\phi)}{\varepsilon^2} \geq \frac{1}{2}\int_{[\phi\neq 0]} a_0\phi^2.$$

Hence we obtain

$$\frac{1}{2}\int_{\Omega} |\nabla\phi|^2 - \int_{\Omega} \frac{F(x,\varepsilon\phi)}{\varepsilon^2} < 0 \quad \text{for} \quad \varepsilon > 0 \quad \text{small enough.}$$

□

Conclusion of Proof. Using (8), (9) and (10) we see that $\inf_{u\in H_0^1} E(u)$ is achieved by some $u \not\equiv 0$. We may always assume that $u \geq 0$. Otherwise we replace u by u^+ and use the fact that $F(x,u) \leq F(x,u^+)$. Here the last inequality holds since $F(x,u) = f(x,0)u \leq 0$ for $u \leq 0$. Then we know that $E(u)$ is of class C^1 . Thus there exists a weak solution u of (P). If we knew in addition that $u \in L^\infty(\Omega)$, we would conclude that u is a solution of (P). To show that $u \in L^\infty(\Omega)$, we introduce a truncated problem. We set, for each integer $k > 0$,

$$\begin{cases} f^k(x,u) = \max\{f(x,u), -ku\} & \text{if } u \geq 0 \\ f^k(x,u) = f^k(x,0) = f(x,0) & \text{if } u \leq 0 \end{cases}$$

and

$$a_0^k(x) = \liminf_{u\downarrow 0} \frac{f^k(x,u)}{u}.$$

Assumptions (H1), (H2) and (H3) hold for $f^k(x,u)$. Since $f \leq f^k$ and $a_0(x) \leq a_0^k(x)$, $\lambda_1(-\bigtriangleup - a_0^k(x)) \leq \lambda_1(-\bigtriangleup - a_0(x)) < 0$ holds. From this, the assumption (1) holds for $a_0^k(x)$. Moreover, the assumption (4) holds for $f^k(x,u)$ provided that k is large enough. Set

$$E_k(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F^k(x,u)$$

for all $u \in H_0^1(\Omega)$. It follows from the previous argument that $\inf_{u\in H_0^1} E_k(u)$ is achieved by some u_k. Moreover, u_k satisfies

$$\begin{cases} -\triangle u_k = f^k(x,u_k) & \text{in } \Omega \\ u_k \geq 0, u_k \not\equiv 0 \text{ in } \Omega \\ u_k = 0 \text{ on } \partial\Omega. \end{cases}$$

Then there exist constants D_k, C_k such that

$$-D_k(|u|+1) \leq f^k(x,u) \leq C_k(|u|^{p-1}+1).$$

Therefore $E_k(u)$ is of class C^1 and by a standard bootstrap argument, $u_k \in L^\infty(\Omega)$. Set $v = \min\{u, u_k\}$. We claim that

$$E(v) \leq E(u).$$

This shows that $u \in L^\infty(\Omega)$. Indeed, we have

$$\frac{1}{2}\int_\Omega |\nabla u_k|^2 - \int_\Omega F^k(x,u_k) \leq \frac{1}{2}\int_\Omega |\nabla \phi|^2 - \int_\Omega F^k(x,\phi)$$

for all $\phi \in H_0^1(\Omega)$. Choosing $\phi = \max\{u, u_k\}$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{[u_k\geq u]} |\nabla u_k|^2 - \int_{[u_k\geq u]} F^k(x,u_k) \\ &+\frac{1}{2}\int_{[u_k<u]} |\nabla u_k|^2 - \int_{[u_k<u]} F^k(x,u_k) \\ &\leq \frac{1}{2}\int_{[u_k\geq u]} |\nabla u_k|^2 - \int_{[u_k\geq u]} F^k(x,u_k) \\ &+\frac{1}{2}\int_{[u_k<u]} |\nabla u|^2 - \int_{[u_k<u]} F^k(x,u). \end{aligned}$$

Thus we find

$$
\begin{aligned}
&\frac{1}{2}\int_{[u_k<u]} |\nabla u_k|^2 - \int_{[u_k<u]} F^k(x,u_k) \\
&\leq \frac{1}{2}\int_{[u_k<u]} |\nabla u|^2 - \int_{[u_k<u]} F^k(x,u).
\end{aligned} \tag{11}
$$

On the other hand, we have

$$
\begin{aligned}
E(v)-E(u) &= \int_{[u_k<u]} \{\frac{1}{2}|\nabla u_k|^2 - \frac{1}{2}|\nabla u|^2 - F(x,u_k) + F(x,u)\} \\
&\leq \int_{[u_k<u]} F^k(x,u_k) - F^k(x,u) - F(x,u_k) + F(x,u) \\
&= \int_{[u_k<u]} [\int_{u_k}^{u} f(x,t) - f^k(x,t)]dt \leq 0.
\end{aligned}
$$

Thus $E(v) \leq E(u)$. From this, we know $v = u$, $u \leq u_k$. Therefore $u \in L^\infty(\Omega)$. □

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