Comm. Korean Math. Soc. **14** (1999), No. 2, pp. 295–299

A NOTE ON CONNECTEDNESS OF QUASI-RANDOM GRAPHS

CHANGWOO LEE

ABSTRACT. Every quasi-random graph $G(n)$ on n vertices consists of a giant component plus $o(n)$ vertices, and every quasi-random graph $G(n)$ with minimum degree $(1+o(1))\frac{n}{2}$ is connected.

1. Introduction and preliminaries

Let us consider the random graph model for graphs with n vertices and edge probability $p = 1/2$. Thus the probability space $\Omega(n)$ consists of all labeled graphs G of order n, and the probability of G is given by $\Pr(G) = 2^{-\binom{n}{2}}$. For a graph property $\mathcal{P}$, it may happen that

$$\Pr\{G \in \Omega(n) \mid G \text{ satisfies } \mathcal{P}\} \to 1 \text{ as } n \to \infty.$$

In this case, a typical graph in $\Omega(n)$, which we denote by $G_{1/2}(n)$, will have property $\mathcal{P}$ with overwhelming probability as n becomes large. We abbreviate this by saying that a random graph $G_{1/2}(n)$ has property $\mathcal{P}$ *almost surely.* For details of these concepts, see [1] or [6].

One would like to construct graphs that behave just like a random graph $G_{1/2}(n)$. Of course, it is logically impossible to construct a truly random graph. Thus Chung, Graham, and Wilson defined in [4] quasi-random graphs, which simulate $G_{1/2}(n)$ without much deviation. Among many equivalent quasi-random properties studied in [4] and [3], we list only three needed in this paper. Let $G(n)$ denote a graph on n vertices. A family $\{G(n)\}$ of graphs (or for brevity, a graph $G = G(n)$) is *quasi-random* if it satisfies any one of and hence all of the following.

$P_1(s)$: For fixed s, each labeled graph $M(s)$ on s vertices occurs $(1 + o(1))n^s/2^{\binom{s}{2}}$ times as an induced subgraph of G.

Received March 26, 1998. Revised January 4, 1999.
1991 Mathematics Subject Classification: 05C80.
Key words and phrases: quasi-random, connected.

P_4: For each subset $S \subseteq V(G)$, the number $e(S)$ of edges in $G[S]$ is $e(S) = \frac{1}{4}|S|^2 + o(n^2)$. Here, $G[S]$ denotes the subgraph of G induced by S.

Q: For each subset $S \subseteq V(G)$, the number $e(S, \overline{S})$ of edges between S and $\overline{S}$ satisfies $e(S, \overline{S}) = \frac{1}{2}|S||\overline{S}| + o(n^2)$, where $\overline{S} = V(G) - S$.

Another property of $G(n)$, which is weaker than quasi-randomness, is the following.

P_0': All but $o(n)$ vertices have degree $(1 + o(1))\frac{n}{2}$. In this case we say that $G(n)$ is *almost-regular.*

Note that the Paley graph Q_p on p vertices is quasi-random ([4]) and strongly regular with parameters $((p-1)/2, (p-5)/4, (p-1)/4)$ ([1]).

In this paper, we show how much quasi-random graphs deviate from $G_{1/2}(n)$ in connectedness. All definitions and notation are the same as in [4] and [3].

Two corollaries of the theorem of Chung, Graham, and Wilson which asserts the equivalence of $P_1(s)$, P_4, and Q are stated next. They follow immediately from property Q.

COROLLARY 1 ([4]). *Let $\epsilon > 0$ and suppose $G = G(n)$ is quasi-random. Then for any $X \subseteq V(G)$ with $|X| > \epsilon n$, the subgraph $G[X]$ of G induced by X is quasi-random.*

COROLLARY 2. *Let $G = G(n)$ be a quasi-random graph. Then the complement $\overline{G}$ of G is also quasi-random.*

From a given quasi-random graph, we can construct another quasi-random graph using the following lemma.

LEMMA. *Let $G = G(n)$ be a graph on n vertices constructed as follows. The vertex set of G consists of two disjoint sets V_1 and V_2 with $|V_1| = o(n)$. On V_1 we place any graph while on V_2 a quasi-random graph. Between V_1 and V_2 we place any bipartite graph. Then the graph G is quasi-random.*

PROOF. We want to show that G satisfies property P_4. Let S be a subset of vertices of G and let $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$. Then we have

$$\begin{aligned}
e(S_1) &= o(n^2) \quad \text{since} \quad 0 \le e(S_1) \le \binom{|S_1|}{2} = o(n^2), \\
e(S_2) &= \frac{1}{4}|S_2|^2 + o(|V_2|^2) = \frac{1}{4}(|S| + o(n))^2 + o(n^2) \\
&= \frac{1}{4}|S|^2 + o(n^2), \\
e(S_1, S_2) &= o(n^2) \quad \text{since} \quad e(S_1, S_2) \le |S_1||S_2| = o(n^2).
\end{aligned}$$

Therefore we have

$$e(S) = e(S_1) + e(S_2) + e(S_1, S_2) = \frac{1}{4}|S|^2 + o(n^2).$$

This completes the proof. □

2. Main results

In this section, we investigate the connectedness of quasi-random graphs. We know that $G_{1/2}(n)$ is connected almost surely. But quasi-random graphs need not be connected as we can see in the case of a quasi-random graph consisting of the union of the Paley graph and an added isolated vertex. For quasi-random graphs, we have the following weaker result.

THEOREM. *Every quasi-random graph $G = G(n)$ on n vertices consists of a giant component and at most $o(n)$ vertices outside the giant.*

PROOF. Let $G = G(n)$ be a quasi-random graph on n vertices and let $H = H(m)$ be the subgraph of G induced by $S = \{v \in V(G) \mid \deg_G(v) \ge (1 + o(1))\frac{n}{2}\}$, where $m = |V(H)|$. Then, since G is almost-regular, $m = (1+o(1))n$ and $\deg_H(v) \ge (1+o(1))\frac{m}{2}$ for all $v \in V(H)$. Moreover, every component of $H(m)$ contains at least $(1+o(1))\frac{m}{2}$ vertices.

First, we want to show that $H(m)$ consists of at most two components of order at least $(1+o(1))\frac{m}{2}$. Let $0 < \epsilon < \frac{1}{4}$. Then there exists a number $m_0 = m_0(\epsilon)$ such that $deg_H(v) \ge (1-\epsilon)\frac{m}{2} > \frac{3m}{8}$ for all $m \ge m_0$ and all

$v \in V(H)$. Suppose that for some $m \geq m_0$, there exist three components L_1, L_2, and L_3 of $H(m)$. Then we have

$$\begin{aligned} m &\geq |V(L_1)| + |V(L_2)| + |V(L_3)| \\ &\geq 3(1-\epsilon)\frac{m}{2} > 3\frac{3m}{8} \\ &= \frac{9m}{8}, \end{aligned}$$

which is a contradiction. Therefore H has at most two components of order at least $(1+o(1))\frac{m}{2}$.

Next, we want to show that $H(m)$ is connected for sufficiently large m. Suppose that $H(m)$ is not connected for infinitely many m. Then for such m's, $H(m)$ consists of two components L_1 and L_2 of order at least $(1+o(1))\frac{m}{2}$. Now it is easy to see that $e(L_1, L_2) = 0$ for infinitely many m. On the other hand, $H(m)$ itself is quasi-random from Corollary 1, and hence, from property Q, we have

$$\begin{aligned} e(L_1, L_2) &= \frac{1}{2}|V(L_1)||V(L_2)| + o(m^2) \\ &\geq \frac{1}{2}(\frac{3m}{8})^2 + o(m^2) \end{aligned}$$

for infinitely many m. This contradicts the fact that $e(L_1, L_2) = 0$ for infinitely many m. Thus $H(m)$ consists of only one component for sufficiently large m, and the theorem follows. □

But if we give degree restrictions to G, then we can conclude that G is connected.

COROLLARY 3. *Let $G = G(n)$ be a quasi-random graph on n vertices. If $\delta(G) = (1+o(1))\frac{n}{2}$, then G is connected.*

EXAMPLES. (1) The Paley graph Q_p on p vertices is quasi-random and $(p-1)/2$-regular. Hence by Corollary 3, both Q_p and the complement $\overline{Q_p}$ are connected for sufficiently large p. Of course, it follows immediately from its definition that Q_p is connected for all p.

(2) Let F_n be a field with n elements and let $AP(F_n)$ be the affine plane of order n. Let S be a subset of "slopes" of the $n+1$ parallel

classes of lines such that $|S| \sim \frac{n}{2}$. We define a graph $G(n^2) = (V, E)$ as follows. Let V be the set of all points in $AP(F_n)$ and let $xy \in E$ iff the slope of the line in $AP(F_n)$ containing x and y belongs to S. Then $G(n^2)$ is a quasi-random graph of order n^2 [4], and every vertex of $G(n^2)$ has degree $(n-1)|S| \sim \frac{n^2}{2}$. Hence by Corollary 3, both $G(n^2)$ and the complement $\overline{G(n^2)}$ are connected for sufficiently large n.

(3) We define a graph $G_n = (V, E)$ as follows. Let V be the set of all n-subsets of a fixed $2n$-set and let $xy \in E$ iff $|x \cap y| \equiv 0 \pmod 2$. Then G_n is a quasi-random graph of order $\binom{2n}{n}$ (see [4] or [2]). Every vertex v of G_n has degree

$$\deg(v) = \begin{cases} \frac{1}{2}\binom{2n}{n} & \text{if } n \text{ is odd} \\ \frac{1}{2}\binom{2n}{n} + \frac{(-1)^{n/2}}{2}\binom{n}{n/2} - 1 & \text{if } n \text{ is even} \end{cases}$$

and hence $\deg(v) \sim \frac{1}{2}\binom{2n}{n}$. Therefore both G_n and the complement $\overline{G_n}$ are connected for sufficiently large n. Of course, it follows immediately from Dirac's theorem ([5]) that G_n is connected when n is odd or when $n/2$ is even. However, it seems to be difficult to show without using quasi-randomness that G_n is connected when $n/2$ is a sufficiently large odd integer.

References

[1] B. Bollobás, *Random Graphs*, Academic, London, 1985.

[2] F. R. K. Chung and R. L. Graham, *Quasi-random Set Systems*, J. Amer. Math. Soc. **4** (1991), 151–196.

[3] ———, *Maximum Cuts and Quasi-random Graphs*, Random Graphs Vol. 2 (A. Frieze and T. Luczak, eds.), Wiley, New York, 1992, pp. 23–33.

[4] F. R. K. Chung, R. L. Graham, and R. M. Wilson, *Quasi-random Graphs*, Combinatorica **9** (1989), 345–362.

[5] V. Chvátal and P. Erdös, *A Note on Hamiltonian Circuits*, Proc. London Math. Soc. **2** (1952), 69–81.

[6] E. M. Palmer, *Graphical Evolution*, Wiley, New York, 1985.

Department of Mathematics
University of Seoul
Seoul 130–743, Korea
E-mail: chlee@uoscc.uos.ac.kr