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A GENERALIZATION OF PREECE'S IDENTITY

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ABSTRACT. The aim of this research is to provide a generalization of the well-known, interesting and useful identity due to Preece by using classical Dixon's theorem on a sum of ${}_3F_2$.

1. Introduction and preliminaries

The generalized hypergeometric function with p numerator and q denominator parameters is defined by

$$\tag{1.1} \begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};z\right] &= {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) \\ &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n z^n}{(\beta_1)_n\ldots(\beta_q)_n n!} \end{aligned}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined by, α any complex number,

$$\tag{1.2} (\alpha)_n := \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1) & \text{if } n \in \mathbf{N} := \{1, 2, 3, \ldots\}, \\ 1 & \text{if } n = 0. \end{cases}$$

Using the fundamental property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

$$\tag{1.3} (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

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where Γ is the well-known Gamma function.

We just introduce some necessary identities related to the Pochhammer symbol and the Gamma function without proof:

(1.4)
$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \quad (0 \leq k \leq n);$$

(1.5)
$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n};$$

(1.6)
$$\Gamma(\alpha - n)\Gamma(1-\alpha+n) = (-1)^n \Gamma(\alpha)\Gamma(1-\alpha);$$

(1.7)
$$(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n \quad (n \in \mathbf{N} \cup \{0\});$$

(1.8)
$$\Gamma(2\alpha) = 2^{2\alpha-1}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right) \Big/ \Gamma\left(\frac{1}{2}\right),$$

which is the well-known Legendre duplication formula for the Gamma function.

From the theory of differential equations, Professor Preece [3] established the following well known, interesting and useful identity involving product of generalized hypergeometric series:

(1.9)
$${}_1F_1(\alpha;\ 2\alpha;\ x) \times {}_1F_1(\alpha;\ 2\alpha;\ -x) = {}_1F_2(\alpha;\ \alpha+\frac{1}{2},\ 2\alpha;\ \frac{x^2}{4}).$$

Later on Professor Bailey [1] generalized this identity and obtained the following very interesting and useful result:

(1.10)
$${}_1F_1(\alpha; 2\alpha; x) \times {}_1F_1(\beta; 2\beta; -x) = {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{1}{2}, \beta+\frac{1}{2}, \alpha+\beta \end{matrix}; \frac{x^2}{4}\right]$$

by using classical Watson's theorem [2] on a sum of ${}_3F_2$:

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})\Gamma(c-\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{b}{2}+\frac{1}{2})} \tag{1.11}
$$

provided $\mathrm{Re}(2c-a-b) > -1$.

We are aiming at giving a generalization of (1.9) by using classical Dixon's theorem [2] on a sum of ${}_3F_2$:

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)} \tag{1.12}
$$

provided $\mathrm{Re}(a-2b-2c) > -2$.

2. A Generalization of (1.9)

We want to show the following generalization of (1.9): For $l \in \mathbf{N}$,

$$
{}_1F_1(\alpha; l\alpha; x) \times {}_1F_1(\alpha; l\alpha; -x) = {}_2F_3\left[\begin{matrix} \alpha, l\alpha - \alpha \\ l\alpha, \frac{1}{2}l\alpha, \frac{1}{2}l\alpha + \frac{1}{2} \end{matrix}; \frac{x^2}{4}\right], \tag{2.1}
$$

which, for the case $l = 2$, reduces immediately to the Preece's identity (1.9).

Indeed, let

$$
{}_1F_1(\alpha; l\alpha; x) \times {}_1F_1(\alpha; l\alpha; -x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}, \tag{A}
$$

since the left-hand side of (A) is an even function of x.

Beginning with the left-hand side of (A), we find

$$\begin{aligned}
&{}_1F_1(\alpha;\ l\alpha;\ x) \times {}_1F_1(\alpha;\ l\alpha;\ -x) \\
&= \left[\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(l\alpha)_n} \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(l\alpha)_n} \frac{(-x)^n}{n!}\right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{(\alpha)_{n-k}(\alpha)_k(-1)^k}{(l\alpha)_{n-k}(l\alpha)_k(n-k)!k!}\right] x^n,
\end{aligned}$$

in which, using the identity (1.4), we have

$$\text{(B)}\qquad \begin{aligned}
a_n &= \frac{(\alpha)_n}{n!(l\alpha)_n} \sum_{k=0}^{n} \frac{(-n)_k(\alpha)_k(1-l\alpha-n)_k}{(1-\alpha-n)_k(l\alpha)_k k!} \\
&= \frac{(\alpha)_n}{n!(l\alpha)_n} {}_3F_2\left[\begin{matrix} -n,\ \alpha,\ 1-l\alpha-n \\ 1-\alpha-n,\ l\alpha \end{matrix};\ 1\right].
\end{aligned}$$

Replacing n by $2n$ in (B), we obtain

$$\text{(C)}\qquad a_{2n} = \frac{(\alpha)_{2n}}{(2n)!(l\alpha)_{2n}} {}_3F_2\left[\begin{matrix} -2n,\ \alpha,\ 1-l\alpha-2n \\ 1-\alpha-2n,\ l\alpha \end{matrix};\ 1\right].$$

To apply Dixon's theorem (1.12) to (C) by setting $a = -2n$, $b = \alpha$ and $c = 1 - l\alpha - 2n$, we have

$$\text{(D)}\qquad \begin{aligned}
a_{2n} &= \frac{(\alpha)_{2n}}{(2n)!(l\alpha)_{2n}} \frac{\Gamma(l\alpha)}{\Gamma(l\alpha+n)} \frac{\Gamma(l\alpha-\alpha+n)}{\Gamma(l\alpha-\alpha)} \cdot \\
&\quad \cdot \frac{\Gamma(1-n)}{\Gamma(1-2n)} \frac{\Gamma(1-2n-\alpha)}{\Gamma(1-n-\alpha)}.
\end{aligned}$$

Now, making use of identities (1.6) and (1.8), we find

$$\text{(E)}\qquad \frac{\Gamma(1-n)}{\Gamma(1-2n)} = (-1)^n \left(\frac{1}{2}\right)_n 2^{2n}.$$

Also using (1.5), we have

$$\text{(F)} \qquad \frac{\Gamma(1-2n-\alpha)}{\Gamma(1-n-\alpha)} = (-1)^n \frac{(\alpha)_n}{(\alpha)_{2n}}.$$

If we put (E) and (F) into (D) with the aid of (1.3) and (1.7), we readily obtain

$$\text{(G)} \qquad a_{2n} = \frac{(\alpha)_n (l\alpha - \alpha)_n}{2^{2n} n! \left(\frac{l\alpha}{2}\right)_n \left(\frac{l\alpha}{2} + \frac{1}{2}\right)_n (l\alpha)_n}.$$

Finally applying (G) to (A) produces immediately our desired result (2.1).

3. Concluding remarks

We conclude by saying that the generalization (2.1) of the Preece's identity (1.9) given in section 2 cannot be derived with the help of classical Watson's theorem.

On the other hand, for another very short proofs of (1.9) and (1.10) and a few other interesting contiguous results, see references [4], [5], [6] and [7].

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