Comm. Korean Math. Soc. **14** (1999), No. 1, pp. 171–177

TOEPLITZ OPERATORS ON BERGMAN SPACES DEFINED ON UPPER PLANES

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Abstract. We study some properties of Toeplitz operators on the Bergman spaces $B^p(H_r)$, where $H_r = \{x + iy : y > r\}$. We consider the pseudo-hyperbolic disk and the covering property. We also obtain some characterizations of compact Toeplitz operators.

1. Introduction

For any $r \in \mathbb{R}$, let $H_r = \{x + iy \in \mathbb{C} : y > r\}$. Then H_r is an upper plane in $\mathbb{C}$ and invariant under horizontal translations. For $1 \le p < \infty$, we define $B^p = \{f \in L^p(H_r) : f$ is a holomorphic function on $H_r\}$ which is called a Bergman space. Some properties on Bergman spaces of the unit disk have been well studied; see [1],[2],[5]. Then a point evaluation is a bounded linear functional on B^p. For each fixed $w \in H_r$, there is a unique function $K(\cdot, w) \in B^2$ such that $f(w) = \langle f, K(\cdot, w)\rangle = \int_{H_r} f(z)\overline{K(z,w)}\,dz$ for all $f \in B^2$. In fact, $K(z,w) = \frac{1}{\pi(2r+(z-\overline{w})i)^2}$.

Since B^2 is a closed subspace of L^2, there is a unique orthogonal projection $P : L^2 \longrightarrow B^2$ defined by $Pf(w) = \int_{H_r} f(z)\overline{K(z,w)}\,dz$ for all $f \in L^2$. Moreover, for any $f \in B^p$, $Pf = f$. If $p = 1$ then the projection P is not bounded but for $1 < p < \infty$, P is bounded and hence we get that the dual of B^p is B^q,where $\frac{1}{p} + \frac{1}{q} = 1$. Much information about Bergman space defined on H_r is in [4].

Received July 16, 1998. Revised September 21, 1998.
1991 Mathematics Subject Classification: Primary 47B35, 47B38; Secondary 31B10, 30D55.
Key words and phrases: Toeplitz operators, Bergman spaces, upper plane, pseudo-hyperbolic disk.

This paper deals with the pseudo-hyperbolic distance and the compactness of Toeplitz operators on $B^p(H_r)$. We find the connection between pseudo-hyperbolic disks and Euclidean disks and the covering property (see [1],[3]). This implies two quantities are equivalent. For $f \in L^\infty(H_r, dA)$, we define $T_f(g) = P(fg)$. Then T_f is bounded. Under some conditions, we show that T_f is compact iff $f \in C_0(H_r)$.

2. Toeplitz operators on B^p

Let $w = x + iy \in H_r$. We define $\varphi_w : H_r \longrightarrow H_r$ by $\varphi_w(z) = \frac{s-x}{y-r} + i(\frac{t-r}{y-r} + r)$, where $z = s+it$. Then φ_w is 1-1, onto and holomorphic on H_r and we define the pseudo-hyperbolic distance $d(w,z)$ by $d(w,z) = \frac{|z-w|}{|z-\tilde{w}|}$, where $\tilde{w} = x+i(2r-y)$ and φ_w preserves pseudo-hyperbolic distance. For $0 < R < 1$ and $w \in H_r$, the pseudo-hyperbolic disk $D(w,R) = \{z \in H_r : d(w,z) < R\}$.

PROPOSITION 2.1. *For $0 < R < 1$ and $w = x + iy \in H_r$, $D(w,R)$ is the Euclidean disk with center $(x, \frac{1+R^2}{1-R^2}(y-r)+r)$ and radius $\frac{2R(y-r)}{1-R^2}$ which is denoted by $B\big((x, \frac{1+R^2}{1-R^2}(y-r)+r), \frac{2R(y-r)}{1-R^2}\big)$ and $\varphi_w\big(D(w,R)\big) = D\big((r+1)i, R\big)$.*

PROOF. Let $\rho(z,w)$ denote the pseudo-hyperbolic distance on H between z and w. Then $z \in D(w,R)$ iff $\rho(z-ri, w-ri) < R$ iff $z - ri \in B\big((x, \frac{1+R^2}{1-R^2}(y-r)), \frac{2R(y-r)}{1-R^2}\big)$ iff $z \in B\big((x, \frac{1+R^2}{1-R^2}(y-r)+r), \frac{2R(y-r)}{1-R^2}\big)$. Moreover, $z \in \varphi_w\big(D(w,R)\big)$ iff $z = \varphi_w(u)$ for some $u \in D(w,R)$. If $u = s+it$ then $\varphi_w(u) = \frac{s-x}{y-r} + i(\frac{t-r}{y-r}+r)$ and hence $z \in \varphi_w\big(D(w,R)\big)$ iff $z \in D\big((r+1)i, R\big)$. □

COROLLARY 2.2. *For $w = x + iy \in H_r$ and $0 < R < 1$, $\big|D(w,R)\big| = \frac{4\pi R^2(y-r)^2}{(1-R^2)^2}$.*

Suppose $w = x+iy$. Since $\varphi_w^{-1}\Big(D\big((r+1)i,R\big)\Big) = D(w,R)$, $\inf\{|K_w(z)| : z \in D(w,R)\} = \inf\{|K_w(z)| : z \in \varphi_w^{-1}\Big(D\big((r+1)i,R\big)\Big)\} = \frac{1}{\pi(y-r)^2} \inf\{\frac{1}{s^2+(t-r+1)^2} : z = s+it \in D\big((r+1)i,R\big)\} = \frac{1}{\pi(y-r)^2}(\frac{1-R}{2})^2$. Then one has the following:

LEMMA 2.3. *If* $w \in H_r$ *and* $0 < R < 1$ *then* $\inf\{|K_w(z)| : z \in D(w,R)\} = \frac{(1-R)^2}{4\pi(y-r)^2}$ *and* $\sup\{|K_w(z)| : z \in D(w,R)\} = \frac{(1+R)^2}{4\pi(y-r)^2}$.

LEMMA 2.4. *Let* $0 < R < t < 1$ *and* $1 \le p < \infty$. *Then for any* $w \in H_r$, $z \in D(w,R)$ *and any holomorphic function* f *on* H_r, *there is* $C > 0$ *such that* $|f(z)|^p \le \frac{C}{|D(w,t)|} \int_{D(w,t)} |f|^p \, dA$.

PROOF. Let f be holomorphic on H_r and let $w \in H_r$. Since $D(w,R) = \varphi_w^{-1}\Big(D\big((r+1)i, R\big)\Big)$, for $z \in D(w,R)$, there is $\lambda \in D\big((r+1)i, R\big)$ such that $z = \varphi_w^{-1}(\lambda)$. Put $R_0 = d\Big(\partial D\big((r+1)i, R\big), \partial D\big((r+1)i, t\big)\Big)$. Then $B\big(\varphi_w(z), R_0\big) \subseteq D\big((r+1)i, t\big)$ and $f(z) = f \circ \varphi_w^{-1}(\lambda) = \frac{1}{\left|B(\varphi_w(z),R_0)\right|} \int_{B\left(\varphi_w(z),R_0\right)} f \circ \varphi_w^{-1} \, dA$. Hence $\left|f(z)\right|^p \le \frac{1}{\pi R_0^2} \int_{B\left(\varphi_w(z),R_0\right)} \left|f \circ \varphi_w^{-1}\right|^p dA \le \frac{1}{\pi R_0^2} \int_{D\left((r+1)i,t\right)} \left|f \circ \varphi_w^{-1}\right|^p dA = \frac{1}{\pi R_0^2} \int_{D(w,t)} \left|f \circ \varphi_w^{-1}\big(\varphi_w(u)\big)\right|^p \left|\varphi_w{}'(u)\right|^2 dA(u)$ $= \frac{1}{\pi R_0^2 (y-r)^2} \int_{D(w,t)} |f|^p \, dA \doteq \frac{C}{\left|D(w,t)\right|} \int_{D(w,t)} |f|^p \, dA$, where $C = \frac{4t^2}{R_0^2(1-t^2)^2}$. $\square$

Let $0 < R < 1$ and let $\{B_n\}$ be a sequence of pseudo-hyperbolic disks in H_r of radius $\frac{R}{3}$ such that $\bigcup_{n=1}^{\infty} B_n = H_r$. Put $D_1 = B_1$. For $n \ge 2$, let i =first$\{k \in \mathbb{N} : B_k \cap \big(\bigcup_{j=1}^{n-1} D_j = \emptyset\}$. Put $D_n = B_i$. Let w_n be the pseudo-hyperbolic center of D_n. Then $\bigcup_{n=1}^{\infty} D(w_n, R) = H_r$. If $z = x + iy \in H_r$ then $\pi R^2 (y-r)^2 < \left|D(z,R)\right| = \frac{4\pi R^2 (y-r)^2}{(1-R^2)^2} < \frac{4\pi R^2 (y-r)^2}{(1-R)^2}$ and hence $R^2\pi < \frac{\left|D(z,R)\right|}{(y-r)^2} < \frac{4\pi R^2}{(1-R)^2}$. Let $J_z = \{m : d(w_m, z) < \frac{2R+1}{3}\}$. Since $\lim_{l\to\infty} \frac{2\cdot 3^{l-1}}{2\cdot 3^{l-1}+1} = 1$, there is $k \in \mathbb{N}$ such that $R < \frac{2\cdot 3^{k-1}}{2\cdot 3^{k-1}+1}$. Then $D(w_n, \frac{R}{3^k}) \subseteq D\big(z, \frac{(2\cdot 3^{k-1}+1)R + 3^{k-1}}{3^k}\big)$ whenever $n \in J_z$. Since for each $m \in J_z$, $\frac{1-\frac{2R+1}{3}}{1+\frac{2R+1}{3}} < \frac{\text{Im}w_m - r}{\text{Im}z - r}$, $\sum_{m\in J_z} \left(\frac{R}{3^k}\right)^2 \pi \big(\text{Im}w_m - r\big)^2 \le \sum_{m \in J_z} \left|D(w_m, \frac{R}{3^k})\right| \le \left|D(z, \frac{(2\cdot 3^{k-1}+1)R+3^{k-1}}{3^k})\right| \le 4\pi \left(\frac{\frac{(2\cdot 3^{k-1}+1)R+3^{k-1}}{3^k}}{1-\frac{(2\cdot 3^{k-1}+1)R+3^{k-1}}{3^k}}\right)^2 (\text{Im}z - r)^2$, and hence $|J_z|$ is bounded by some constant which is depend on k and R. Summarizing this observation:

LEMMA 2.5. *For* $0 < R < 1$, *there is a sequence* $\{w_n\}$ *in* H_r *and a positive integer* M *such that* $\bigcup_{n=1}^{\infty} D(w_n, R) = H_r$ *and for each* $z \in H_r$, $\left|\{n : z \in D(w_n, \frac{2R+1}{3})\}\right| \le M$.

Proposition 2.6. *Let $0 < R < 1$, $1 \leq p < \infty$ and μ a positive Borel measure on H_r. Then there are constants C and D such that* $\sup_{w\in H_r} \frac{\mu\left(D(w,R)\right)}{\left|D(w,R)\right|} \leq C \sup_{\substack{f\in B^p\\ f\neq 0}} \frac{\int_{H_r}|f|^p\,d\mu}{\int_{H_r}|f|^p\,dA} \leq D \sup_{w\in H_r} \frac{\mu\left(D(w,R)\right)}{\left|D(w,R)\right|}$.

Proof. Take any $w = x + iy \in H_r$. If $g(z) = \frac{1}{\pi^{\frac{2}{p}}\left(2r+(z-\overline{w})i\right)^{\frac{4}{p}}}$ then $g \in B^p$ and $\int_{H_r} |g(z)|^p\, dA(z) = \frac{1}{\pi^2}\int_{H_r} \frac{1}{|2r+(z-\overline{w})i|^4}\, dA(z) = \frac{1}{4\pi(y-r)^2}$ and hence $\int_{H_r} |g|^p\, d\mu \geq \int_{D(w,R)} |g|^p\, d\mu \geq \inf\{|K_w(z)|^2 : z \in D(w,R)\}\mu\big(D(w,R)\big) = \left(\frac{(1-R)^2}{4\pi(y-r)^2}\right)^2 \mu\big(D(w,R)\big)$. This implies that $\frac{\int_{H_r}|g|^p\,d\mu}{\int_{H_r}|g|^p\,dA} \geq C' \frac{\mu\left(D(w,R)\right)}{|D(w,R)|}$ for some C'. Let $\{w_n\}$ be the sequence in Lemma 2.5 and let $f \in B^p$ be such that $f \neq 0$. Then

$$\begin{aligned}
\int_{H_r} |f|^p\, d\mu &\leq \sum_{n=1}^{\infty} \int_{D(w_n,R)} |f|^p\, d\mu \\
&\leq \sum_{n=1}^{\infty} \sup_{z\in D(w_n,R)} |f(z)|^p \mu\big(D(w_n,R)\big) \\
&\leq C \sum_{n=1}^{\infty} \frac{\mu\big(D(w_n,R)\big)}{\left|D(w_n, \frac{2R+1}{3})\right|} \int_{D(w_n,\frac{2R+1}{3})} |f|^p\, dA \\
&\qquad \text{for some } C \text{ by Lemma 2.4} \\
&\leq C \sum_{n=1}^{\infty} \sup_{w\in H_r} \frac{\mu\big(D(w,R)\big)}{|D(w,R)|} \int_{D(w_n,\frac{2R+1}{3})} |f|^p\, dA \\
&\leq CM \sup_{w\in H_r} \frac{\mu\big(D(w,R)\big)}{|D(w,R)|} \int_{H_r} |f|^p\, dA \\
&\qquad \text{for some } M \text{ by Lemma 2.5.}
\end{aligned}$$

□

Let $P : L^2(H_r, dA) \longrightarrow B^2$ be the orthogonal projection. For $f \in L^\infty(H_r, dA)$, we define $T_f : B^2 \longrightarrow B^2$ by $T_f(g) = P(fg)$ for all $g \in B^2$. Then $\|T_f(g)\|^2 = \int_{H_r} |T_f(g)|^2\, dA \leq \|f\|_\infty^2 \|g\|_2^2$ and hence T_f is bounded.

Proposition 2.7. *Let K be a compact subset of H_r and let $f \in L^\infty(H_r, dA)$ be such that $f = 0$ on $H_r \setminus K$. Then T_f is compact.*

Proof. Let $\{g_n\}$ be a norm bounded sequence in B^2. Then for each $w = x + iy \in K$, $|g_n(w)| = \left| \int_{H_r} g_n(z)\overline{K_w(z)}\ dA(z)\right| \leq \|g_n\|_2 \|K_w\|_2 \leq$

$\frac{\|g_n\|_2}{2\sqrt{\pi}d(\partial K, \partial H_r)}$ and hence $\{g_n\}$ is a normal family. Then there is a subsequence $\{g_{n_k}\}$ which converges uniformly on K to a holomorphic function g. Since $\int_{H_r} |g_{n_k} f - gf|^2\, dA = \int_K |g_{n_k} - g|^2 |f|^2\, dA \le \|f\|_\infty^2 \|g_{n_k} - g\|_2^2$, $\{T_f(g_{n_k})\}$ converges to $P(gf)$ in B^2. Thus T_f is a compact operator. □

LEMMA 2.8. *$B^2 \cap H^\infty$ is dense in B^2.*

PROOF. Since $C_C(H_r)$ is dense in $L^2(H_r)$, for each $f \in B^2$ and $\varepsilon > 0$, there is $g \in C_C(H_r)$ such that $\|g - f\|_2 < \varepsilon$. For each $\delta > 0$, let $f_\delta(z) = f(z + i\delta)$. Then $f_\delta \in B^2$, $\|f_\delta - f\|_2 \le \|f_\delta - g_\delta\|_2 + \|g_\delta - g\|_2 + \|g - f\|_2$ and f_δ is bounded. Since $\lim_{\delta\to 0} \|f_\delta - g_\delta\|_2 = 0$, $\lim_{\delta \to 0} \|f_\delta - f\|_2 = 0$ and hence $B^2 \cap H^\infty$ is dense in B^2. □

PROPOSITION 2.9. *$\frac{K_w}{\|K_w\|_2}$ tends weakly to 0 in B^2 as $\mathrm{Im} w \to r$.*

PROOF. For any $f \in B^2$, $\langle f, \frac{K_w}{\|K_w\|_2}\rangle = \frac{1}{\|K_w\|_2} f(w) = 2\sqrt{\pi}(y - r) f(w)$, where $w = x + iy \in H_r$. Since $B^2 \cap H^\infty$ is dense in B^2, $\lim_{\mathrm{Im} w \to r} \langle f, \frac{K_w}{\|K_w\|_2}\rangle = 0$. □

THEOREM 2.10. *Let f be a nonnegative function in $L^\infty(H_r, dA)$. Then the following are equivalent:*

(1) *T_f is compact*
(2) *for any $R \in (0,1)$, $\frac{1}{|D(w,R)|}\int_{D(w,R)} f\, dA \to 0$ as $\mathrm{Im} w \to r$*
(3) *there is $R \in (0,1)$ such that $\frac{1}{|D(w,R)|}\int_{D(w,R)} f\, dA \to 0$ as $\mathrm{Im} w \to r$.*

PROOF. For any $R \in (0,1)$, $\frac{1}{|D(w,R)|}\int_{D(w,R)} f\, dA = \frac{(1-R^2)^2}{4\pi R^2 (y-r)^2}\int_{D(w,R)} f\, dA$ $= \inf_{z \in D(w,R)} \frac{|K_w(z)|^2}{\|K_w\|_2^2} \frac{(1-R^2)^2}{(1-R)^4 R^2}\int_{D(w,R)} f\, dA \le \frac{(1+R)^2}{R^2(1-R)^2}\int_{D(w,R)} f \frac{|K_w(z)|^2}{\|K_w\|_2^2}\, dA \le$ $\frac{(1+R)^2}{R^2(1-R)^2}\left\langle \frac{T_f K_w}{\|K_w\|_2}, \frac{K_w}{\|K_W\|_2}\right\rangle$. Then $\frac{1}{|D(w,R)|}\int_{D(w,R)} f\, dA \to 0$ as $\mathrm{Im} w \to r$. Hence one has (2). Clearly (2) implies (3). Suppose (3) is true. For each $n \in \mathbb{N}$, let $K_n = \{(x, y) \in \mathbb{C} : -n \le x \le n, \text{ and } \frac{1}{n} + r \le y \le n + r\}$. Then $T_{f\cdot\chi_{K_n}}$ is compact and $\|T_f - T_{f\cdot\chi_{K_n}}\|^2 \le \sup_{\|g\|_2 = 1}\int_{H_r \setminus K_n} f^2 |g|^2\, dA \le C \sup_{w \in H_r} \frac{1}{|D(w,R)|}\int_{(H_r\setminus K_n)\cap D(w,R)} f^2\, dA$ and hence $\lim_{n\to\infty}\|T_f - T_{f\cdot\chi_{K_n}}\| = 0$. Since $T_{f\cdot\chi_{K_n}}$ is compact, T_f is compact. □

Let H^∞ be the set of all bounded holomorphic functions on H_r and let $\mathcal{U}$ be the norm closed subalgebra of $L^\infty(H_r, dA)$ generated by H^∞ and the complex conjugates of all the functions in H^∞. For any $f = u + iv \in H^\infty$, u and v are bounded harmonic functions. If u is a real-valued bounded harmonic function on H_r then $e^{u + i\tilde{u}} \in H^\infty$. Since $0 \notin \{e^{2u(z)} : z \in H_r\}$,

$loge^{2u} = 2u \in \mathcal{U}$ and hence $\mathcal{U}$ is the norm closed subalgebra of $L^\infty(H_r, dA)$ generated by bounded harmonic functions. Let $\Phi(z) = \frac{f(z)-f(w)}{(z-w)(z-\bar{w})^{-1}}$. Then Φ is holomorphic on H_r and $|\Phi(z)| \leq 2\|f\|_\infty$ for all $z \in H_r$. Thus one has the following:

LEMMA 2.11. *For $f \in H^\infty$, $|f(w) - f(z)| \leq 2\|f\|_\infty d(w, z)$.*

LEMMA 2.12. *For $R \in (0,1)$ and $w \in H_r$, $\int_{D(w,R)} |K_w|^2 \, dA = R^2 \|K_w\|_2^2$.*

PROOF. For $R \in (0,1)$ and $w = x + iy \in H_r$, $|K_w(z)| = \frac{|\rho_w'(z)|}{2\pi(y-r)}$, where $\rho_w(z) = \frac{z-w}{z-\bar{w}}$ and hence $\int_{D(w,R)} |K_w(z)|^2 \, dA(z) = \frac{1}{4\pi^2(y-r)^2} \int_{D(w,R)} |\rho_w'(z)|^2 \, dA(z) = \frac{1}{4\pi^2(y-r)^2} |B(0,R)| = \frac{R^2}{4\pi(y-r)^2} = R^2 \|K_w\|_2^2$. □

Suppose $f \in L^\infty(H_r, dA)$, $g \in H^\infty$ and $h \in B^2$. Since $gh \in B^2$, $T_f T_g(h) = T_f(P(gh)) = T_f(gh) = P(fgh) = T_{fg}(h)$ and hence $T_f T_g = T_{fg}$. Since $\int_{D(w,R)} \int_H |g(u)^2 K_w(u) K_u(z) \overline{K_w(z)}| \, dz du \leq \|g\|_\infty^2 \|K_w\|_2 \int_{D(w,R)} |K_w(u)| \|K_u\|_2 \, du < \infty$, one has the following property:

THEOREM 2.13. *Suppose $f \in H^\infty$ and $\lim_{z \to \infty} f(z) = 0$. Then T_f is compact iff $f \in C_0(H_r)$.*

PROOF. By Proposition 2.7, if $f \in C_0(H_r)$ then T_f is compact. Suppose that there is $\delta > 0$ and a sequence $\{w_n\}$ in H_r such that $\lim_{n \to \infty} \mathrm{Im} w_n = r$ and $|f(w_n)|^2 \geq \frac{\delta}{2}$ for all n. Then there is $R > 0$ such that $|f(z)|^2 \geq \frac{\delta}{2}$ for all $z \in D(w_n, R)$ and $\left\langle \frac{T_{|f|^2} K_{w_n}}{\|K_{w_n}\|_2}, \frac{K_{w_n}}{\|K_{w_n}\|_2} \right\rangle \geq \int_{D(w_n,R)} \frac{|f|^2 |K_{w_n}|^2}{\|K_{w_n}\|_2^2} \, dA \geq \frac{\delta}{2} \int_{D(w_n,R)} \frac{|K_{w_n}|^2}{\|K_{w_n}\|_2^2} \, dA = \frac{\delta}{2} R^2$. This contradicts to the fact that $T_{|f|^2}$ is compact and Proposition 2.9. □

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