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A SIMPLE ALGEBRA GENERATED BY INFINITE ISOMETRIES AND REPRESENTATIONS

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Abstract. We consider the C^*-algebra O_∞ generated by infinite isometries $s_1, s_2, \cdots$ on Hilbert spaces with the property $\sum_{i=1}^n s_i s_i^* \leqq 1$ for every $n \in \mathbb{N}$. We present certain type of representations of C^*-algebra O_∞ on a separable Hilbert space and study the conditions for irreducibility of these representations.

1. Introduction

Joachim Cuntz showed in 1977 that for each $N = 2, 3, \cdots, \infty$ the C^*-algebra from a system N orthogonal isometries, forming a partition of 1, is separable and simple [4]. The existence of such infinite simple C^*-algebra was shown by J. Dixmier in 1964 [6]. It is now called the Cuntz algebra. Recall that the Cuntz algebra $O_N, N = 2, 3, \cdots$ is the C^*-algebra generated by isometries $s_1, s_2, \cdots, s_N$, satisfying

$$s_i^* s_j = \delta_{ij} 1 \quad \text{and} \quad \sum_{i=1}^{N} s_i s_i^* = 1 \tag{1.1}$$

for $i, j \in \{1, \cdots, N\}$. The C^*-algebra O_∞ is the C^*-algebra generated by isometries $s_1, s_2, \cdots$, satisfying

$$s_i^* s_j = \delta_{ij} 1 \quad \text{and} \quad \sum_{i=1}^{n} s_i s_i^* \leqq 1, \tag{1.2}$$

for every $n \in \mathbb{N}$ and $i, j \in \mathbb{N}$. These C^*-algebras are the famous examples whose representations are "bad". A special type of representations of O_N, N for finite, on separable Hilbert spaces $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{T}^n)$,

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$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, n = 1.2, \cdots$, are known (e.g., [2], [3], [8],$\cdots$). In a connection between multiresolution wavelet theory of scale N and representation of the Cuntz algebra O_N, N for finite, the study of the decomposition of representations, so called permutative representations (see Definition 2.1), are recently developed. See [5] for multiresolution analysis from wavelets. Decomposition of this type of representations of finitely generated C^*-algebra has applications to

1. filter functions for multiresolutions from wavelet theory
2. limit problems in analytic number theory
3. multiplicity problems from noncommutative harmonic analysis.

See [2], [3], [4], and [7] in the references. However, representations of O_∞ on a separable Hilbert space are not known in a particularly explicit form. We will present permutative representations of the Cuntz algebra O_∞, which is similar to N for finite, to see applications of the permutative representations of O_∞ in multiresolution analysis from wavelets, and find the conditions for irreduciblity for these representations. This problem is still left open in general. We will present examples; Example 3.6 for irreducible permutative representation, Example 3.7 for infinite irreducible decomposition. We will also present that the answer is negative by showing the set Ω is not a measurable set, for application to multiresolution analysis from wavelets under special condition, but we rather get an example of infinitesimal coming from the permutative representation of the Cuntz algebra O_∞ on $L^2(\mathbb{R})$. See the chapter 4 in this paper. Unlike the case N for finite which has only finite irreducible subrepresentations coming (2.9), the permutative representation of the C^*-algebra O_∞ are decomposed into either finite or infinite irreducible subrepresentations. Of course, the C^*-algebra O_∞ is also simple and separable [4].

2. Permutative representations

We will say that ϕ is a non-degenerated representation of C^*-algebra O_∞, with a slight abuse of terminology, if a representation ϕ satisfies the condition $\sum_{k\in Z} \phi(s_k s_k^*) = 1$, where the sum is in the strong operator topology.

DEFINITION 2.1. A representation of the C^*-algebra O_∞ on a separable Hilbert space $\mathcal{H}$ is said to be *permutative* if there is an orthonormal basis $e_x, x \in X$ for $\mathcal{H}$ such that

$$\phi(s_k)e_x \in \{e_n : n \in X\}, \tag{2.1}$$

where X is a generic countable set for an orthonormal basis with $k \in Z$ and $x \in X$.

Since the Cuntz relations for O_∞ on a separable Hilbert space are equivalently described as the C^*-algebra generated by infinitely many isometries whose ranges are mutually disjoint and the union of these ranges is the whole space if we assume that $\sum_{k\in Z} \phi(s_k s_k^*) = 1$ with the sum in the strong operator topology. From the definition of the permutative representation, we can derive a function system $\sigma_k, k \in Z$ for C^*-algebra O_∞ coming from

$$\phi(s_k)e_x = e_{\sigma_{k(x)}} \tag{2.2}$$

where σ_k are the maps from index set X of an orthonormal basis for a separable Hilbert space into itself. The Cuntz relations in (1.2) imply that

$$\sigma_k : X \longmapsto X \text{ is injective, } k \in \mathbb{Z} \tag{2.3.a}$$

$$\sigma_k(X) \cap \sigma_l(X) = \phi \text{ for } k \neq l \tag{2.3.b}$$

$$\cup_{k\in\mathbb{Z}}\sigma_k(X) = X \tag{2.3.c}$$

where $l \in \mathbb{Z}$. We can modify these formulas with N for finite instead of $\mathbb{Z}$. Conversely, if the maps σ_k, $k \in Z$, satisfy conditions (2.3.a), (2.3.b), and (2.3.c) one can verify that the operators $\phi(s_k)$ on a Hilbert space $\mathcal{H}$ defined by (2.2) satisfy the Cuntz relations (1.2) N for infinite. In fact, the condition (2.3.a) is coming from the definition of the isometrie s_k, the condition (2.3.b) is associated with the property $s_i^* s_j = \delta_{ij}1$, and the condition (2.3.c) is associated with $\sum_{k\in Z} \phi(s_k s_k^*) = 1$. Our task finding a permutative representation of the Cuntz algebra O_∞ on a separable Hilbert space $\mathcal{H}$ thus reduces to finding a countable index set X for an orthonormal basis of the Hilbert space and a system of functions $\sigma_k : X \longmapsto X,\ k \in \mathbb{Z}$, satisfying (2.3.a), (2.3.b), and (2.3.c). With a system of functions $\sigma_k : X \longmapsto X,\ k \in \mathbb{Z}$, on X satisfying (2.3.a), (2.3.b), and (2.3.c), we define an infinite to one map R (which is a joint left inverse of functions σ_k for every $k \in Z$) satisfying

$$R \circ \sigma_k = id_X, \tag{2.4}$$

for every $k \in \mathbb{Z}$. This joint left inverse R of the functions $\sigma_k, k \in \mathbb{Z}$, can be described as following and denoted by σ; if $x \in X$, $\sigma(x) = (j_1, j_2, \cdots)$ defined inductively ; j_1 is the unique $j \in Z$ such that $\phi(s_j^*)e_x \neq 0$ and then $\phi(s_j^*)e_x \in \{e_y : y \in X\}$. When $j_1, j_2, \cdots, j_{k-1}$ are defined inductively, let j_k be the unique $j \in Z$ such that

$$\phi(s_j^*)\phi(s_{j-1}^*)\cdots\phi(s_1^*)e_x \neq 0, \tag{2.5}$$

or to say exactly $R^k(x) = j_k$ in $\sigma(x) = (j_1, j_2, \cdots)$, for nonnegative integer k. We will use both to mean joint left inverse of σ_k through out of this paper.

DEFINITION 2.2. (From [2]) The joint left inverse function $\sigma : X \longmapsto \mathbb{Z}^\infty$ with a function system $\sigma_k,\ k \in \mathbb{Z}$ is called the *coding map* of the function system $\sigma_k \in \mathbb{Z}$. We say that a function system $\sigma_k,\ k \in \mathbb{Z}$ is *multiplicity free* if the coding map σ is injective. We say that the coding map σ is *partially injective* if it satisfies the condition that if $x \in X,\ i_1, \cdots, i_k \in \mathbb{Z}$ and $\sigma(x) = \sigma(\sigma_{i_1}\cdots\sigma_{i_k}(x))$, then $x = \sigma_{i_1}\cdots\sigma_{i_k}(x)$, and the function system is then said to be regular.

We now define an equivalence relation $\sim$ on the index set X; we say that $x \sim y$ if there exist nonnegative integers k_1 and k_2 such that

$$x = \sigma_{i_{k1}}\cdots\sigma_{i_2}\sigma_{i_1}R^{k_2}(y) \tag{2.6}$$

for $x, y \in X, i_1, \cdots, i_{k1} \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_+$. The formula (2.6) can be rewritten as

$$e_x = \phi(s_{i_{k1}})\cdots\phi(s_{i_1})\phi(s_{j_{k_2}}^*)\phi(s_{j_{k_2-1}}^*)\cdots\phi(s_{j_1^*})e_y, \tag{2.7}$$

for $x, y \in X$, $i_1, \cdots, i_{k1}, j_1, \cdots, j_{k_2} \in \mathbb{Z}$.

THEOREM 2.3. *Consider a permutative representation of the Cuntz algebra O_N, $N \in \mathbb{N} \cup \{\infty\}$, on a separable Hilbert space $\mathcal{H}$. Then the closure of the subspace of $\mathcal{H}$ spanned by the vectors e_x, where x runs through a $\sim$ equivalence class, is an irreducible O_∞-module, if the function system is regular. Furthermore, if the functions $\sigma_k, k \in \mathbb{Z}$, is multiplicity free in the sense of* Definition 2.2, *all the modules corresponding to different equivalence classes are unitary inequivalent.*

PROOF. See the reference. (Theorem 2.7 in [2]) □

Let $\mathcal{H}_x$ denote the closure of the subspace of $\mathcal{H}$ spanned by vectors e_x, where x runs through the $\sim$ equivalence class. The restriction of the permutative representation ϕ on $\mathcal{H}_x$ is an irreducible representation of O_N on the separable space $\mathcal{H}_x$, both N for finite or infinite. For a permutative representation of O_∞ on a separable space $\mathcal{H}_x$, suppose that the Hilbert space $\mathcal{H}$ is split into subspaces $\mathcal{H}_{x_i}$ and the representation ϕ is split into subrepresentations $\phi_i = \phi|_{\mathcal{H}_{x_i}}$ of it, for $i \in J$ for some index set $J \subset \mathbb{Z}$, then we have of forms

$$\phi = \bigoplus_{i \in J} \phi_i \quad \text{and} \quad \mathcal{H} = \bigoplus_{i \in J} \mathcal{H}_i. \tag{2.8}$$

Needless to say, the subrepresentations ϕ_i, $i \in J$ are the irreducible subrepresentations of the Cuntz Algebra O_∞ on the Hilbert space $\mathcal{H}_{x_i}$, respectively. We now consider a transcendental number such as π, $e = lim_{n\to\infty}(1 + 1/n)^n$ to construct suitable countable index set X of an orthonormal basis for a separable Hilbert spaces $\mathcal{H}$ so that we can have a permutative representation which is induced from a function system σ_k on X. We will use w for generic transcendental number which is strictly bigger than 1. Let X be a set of polynomials in w with coefficients from the integer set $\mathbb{Z}$ and digit set D a completely incongruent set modulo w in X. See Example 3.6, Example 3.7, Example 4.1, and Example 4.2 for the sets X and D. We now define a function system σ_k on the countable set X described above by

$$\sigma_k(x) = wx + d_k \tag{2.9}$$

where $d_k = k$ modulo w for $k \in \mathbb{Z}$. Then one can check that the functions σ_k on X satisfy the Cuntz related conditions (2.3.a), (2.3.b), and (2.3.c).

N for finite, we define $\sigma_k(x) = Nx + d_k$ for the representation of O_N, where $\{d_1, \cdots, d_N\}$ is a residue set modulo N in $\mathbb{Z}$. The associated representation of the C^*-algebra O_N coming from $\sigma_k(x) = Nx + d_k$ on the Hilbert space $L^2(X)$ is given by

$$\phi(s_k)\xi(x) = x^{d_k}\xi(x^N), \tag{2.10}$$

where $x \in X$, and $\xi \in L^2(X)$. This type of representations of the Cuntz algebra O_N, N for finite, originating from the multiresolution wavelet theory with $X = \mathbb{R}^n$ or $\mathbb{T}^n, n = 1, 2, \cdots$, are introduced if we consider X is an index set of an orthonormal basis for these Hilbert spaces $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{T}^n), n = 1, 2, \cdots$. See [2], [3], [5], and [8].

THEOREM 2.4. *The representations of the Cuntz algebra O_∞ coming* (2.9) *on $L^2(X)$ is regular, thus the representation of O_∞ coming from* (2.9) *is completely split into irreducible subrepresentations.*

PROOF. With Theorem 2.3, the only thing that we need to show is the function system $\sigma_k, k \in \mathbb{Z}$ coming (2.9) is regular. However, it can be showed easily by an elementary calculation with a completely mutually incongruent set D. □

The following questions arise immediately:

Q_1. What are the conditions for the irreducibility of the representation ϕ derived from (2.9) if exist.

Q_2. What are the conditions for finiteness, in the sense of (2.8), of the decomposition of the representation ϕ derived from (2.9) if exist.

Q_3. Is there a representation ϕ derived from (2.9) having infinitely many subrepresentations.

Q_4. Are there application of these representation to multiresolution analysis from wavelets.

We present some results for these questions.

3. Decomposition of the representations

Let D be a residue set in X modulo w for a digit set. $D = \mathbb{Z}$ is an example for obvious case or see the examples in this paper. Define

$$D_M = \sup\{\deg(d) : d \in D\} \tag{3.1.a}$$

$$D_m = \min\{\deg(d) : d \in D\}. \tag{3.1.b}$$

where deg(d):=the degree of d in the form of polynomial in w.

LEMMA 3.1. *Let $x \in X$ and $D_M < \infty$. Then*

$$\deg(R(x)) = -1 + \deg(x), \quad \text{if } D_M < \deg(x) \text{ and } 0 < \deg(x), \tag{3.2.a}$$

$$D_m - 1 \leqq \deg(R(x)), \quad \text{if } \deg(x) < D_m, \tag{3.2.b}$$

loosely speaking

$$\deg(R(x)) < \deg(x), \quad \text{if } D_M < \deg(x) \tag{3.2.a$'$}$$

$$\deg(x) < \deg(R(x)), \quad \text{if } 0 < \deg(x) < D_m. \tag{3.2.b$'$}$$

PROOF. Let $x \in X$ with $\deg(x) = l$, to say $x := a_0 + a_1 w + \cdots + a_l w^l$ for some $a_0, a_1, \cdots, a_l \in \mathbb{Z}$ and $a_l \neq 0$. We first assume $D_M < \deg(x)$ and $0 < \deg(x)$. There then exists a unique d_x such that $x = d_x$ modulo w. Let $d_x = b_0 + b_1 w + \cdots + b_v w^v$ with $v < l$, and $b_1, b_2, \cdots, b_v \in \mathbb{Z}$, note that $a_0 = b_0$ if and only if $x = d_x$ modulo w, then we have

$$\begin{aligned} R(x) &= \frac{x - d_x}{w} \\ &= z_1 + z_2 w + \cdots + z_l w^{l-1} \end{aligned} \tag{3.3}$$

where $z_i = a_i - b_i$ for $1 \leqq i <\leqq v$ and $z_i = a_i$ for $v < i \leqq l - 1$. Since $v \leqq D_M < \deg(x) = l$ and $1 \leqq l$, we have $\deg(R(x)) = -1 + \deg(x)$. For $D_m - 1 \leqq \deg(R(x))$, if $\deg(x) < D_m$ we can assume that $\deg(x) < D_m$ and $0 < D_m$. In this case, we have

$$R(x) = z_1 + z_2 w + \cdots + z_l w^{v-1}$$

where $z_i = a_i - b_i$ for $1 \leqq i \leqq \deg(x)$ and $z_i = -b_i$ for $\deg(x) < i \leqq v - 1$. Hence we have $\deg(R(x)) = -1 + \deg(d_x) \geqq D_m - 1$. □

The formula (3.2.a) and (3.2.b) ensure that the sequence

$$\{\deg(x), \deg(R(x)), \deg(R^2(x)), \cdots\}$$

is decreasing as long as $\deg(R^k(x)) > D_M$, similarly it is increasing as long as $\deg(R^k(x)) < D_m$.

COROLLARY 3.2. *If D_N is finite, then there exists a positive integer n_x for each x such that*

$$D_m - 1 \leqq \deg(R^{n_x}(x)) < D_M \tag{3.4}$$

if $1 \leqq D_m$.

PROOF. The proof follows immediately from Theorem 2.3 and the proof of Lemma 3.1. □

For each $d_k \in D$, there exists unique integer n_k such that $d_k = n_k$ modulo w in X. With $c_k = d_k - n_k$ and $C = \{c_k : k \in \mathbb{Z}\}$, define a set T_∞ in $\mathbb{R}$ and a subset $PreF_\infty$ in $\mathbb{R}$ by

$$T_\infty = \left\{ \sum_{i=1}^{\infty} w^{-i} c_{k_i} : c_{k_i} \in C \right\} \tag{3.5}$$

and

$$PreF_\infty = -T_\infty \cap \{x \in X : D_m - 1 \leqq \deg(x) \leqq D_M - 1\} \tag{3.6}$$

for $D_m \geqq 1$.

$$\begin{aligned} F_\infty := \{ & x \in X : D_m - 1 \leqq \deg(x) \leqq D_M - 1, \\ & \text{coefficients of } x \text{ are of the form} \\ & - c_{i1} - c_{i2} - \cdots - c_t,\ t \leqq D_M - 1 \text{ and } c_{ij} \in Ce(D)\} \end{aligned}$$

LEMMA 3.3. *For each $x \in X$ there exists a positive integer m_x such that*

$$R^{m_x}(x) \in PreF_\infty.$$

PROOF. Let n_x be the least positive integer satisfying $D_m - 1 \leqq \deg(R^{n_x}(x)) \leqq D_M - 1$ and denote $R^{n_x}(x) = a_0 + a_1 w + \cdots + a_t w^t$ for some $D_m - 1 \leqq t \leqq D_M - 1$. By the definition of the joint left inverse R we have

$$R^{n_x+1}(x) = a_1 + a_2 w + \cdots + a_t w^{t-1} - c_{i_0} w^{-1}$$

$$\cdots\cdots\cdots\cdots$$

$$R^{n_x+(t+1)}(x) = - c_{i_0} w^{-(t+1)} - c_{i_1} w^{-t} - \cdots - c_{i_{t-1}} w^{-1} - c_{i_t}$$

where $d_i = a_i$ modulo w and $d_i = c_i + a_i$ for $i = i_0, \cdots, i_t$. Hence $R^{m_x}(x) \in PreF_\infty$ for some large positive integer m_x. □

As we described above, there is an one to one and onto map $f : \mathbb{Z} \mapsto D$ naturally defined by $f(n) = d_n$ satisfying

$$n = d_n \text{ modulo } w.$$

We can rewrite $d_n \in D$ of the form

$$d_n = n + P_{d_n}$$

where $P_{d_n}$is the corresponding polynomial in w obtained by eliminating constant term of d_n in the form of polynomial in w. Let the set $Ce(D)$ denote the set of all nonzero coefficients of the polynomial P_{d_n} for $d_n \in D$.. The set $Ce(D)$ is possibly infinite which is also one of the major differences between permutative representations of the Cuntz algebra O_N, N for finite and the ones of the Cuntz algebra O_∞.

THEOREM 3.4. *If $Ce(D)$ is a finite set and D_M is a finite integer, then for every $x \in X$ the sequence $\{R^l(x)\}_{l=0}^{\infty}$ is eventually periodic in a finite set, i.e., the permutative representations defined from (2.9) are split into finite irreducible subrepresentations.*

PROOF. We can see easily from the proof of Lemma 3.3, the both constant terms of x and d_x in action $R(x)$ disappear. For $k = n_x + (l+1)$, n_x is the number described in Corollary 3.2.

$$R^k(x) = -c_t - c_{t-1}w^{-1} - \cdots c_0 w^{-(t+1)}$$

with $c_i \in C$. $R^k(x)$ can be rewritten as following.

$$R^k(x) = a_0 + a_1 w + \cdots + a_l w^l$$

for some $D_m - 1 \leqq l \leqq D_M - 1$ with $a_i = -c_{i1} - c_{i2} - \cdots - c_{i(D_M-1)}$ for some $c_{ij} \in Ce(D) \cup \{0\}, i = 0, 1, \cdots, l$ and $j = 1, 2, \cdots, D_M - 1$. Therefore we have

$$R^s(x) \in F_\infty$$

for s large enough, where F_∞ is a subset of $PreF_\infty$ as defined. Thus the cardinality of the set F_∞ is at most $(\text{card}(Ce(D))^P$, where the number P is D_M from $(D_M - 1) + 1$. Hence the sequence $\{R^l(x)\}_{l=0}^\infty$ is eventually periodic in the finite set F_∞. The second statement follows from following argument; for every $x \in X$ the sequence $\{R^l(x)\}_{l=0}^\infty$ is eventually periodic in F_∞ which implies $x \sim y$ for some $y \in F_\infty$. Thus the number of inequivalent classes are finite which is at most the cardinality of the set F_∞. By Theorem 2.3, the permutative representation coming from (2.9) is decomposed into finite subrepresentations. □

Remark. The sequence $\{R^l(x)\}_{l=0}^\infty$ is eventually periodic for every $x \in X$ does not imply the finite decomposition of the permutative representation induced by (2.9). See the example 3.7.

Example 3.6 (for irreducible representation). Let l be a fixed positive integer and a digit set $D = \{e^l + z : z \in \mathbb{Z}\}$. Since the only coefficient except constant term is 1, the set $Ce(D)$ is a singleton set $\{1\}$. Furthermore we have $D_m - 1 = D_M - 1 = l - 1$. By Theorem 3.4, we have the singleton set $PreF_\infty = F_\infty = \{-1 - w - \cdots - w^{l-1}\}$. Therefore, for every $x \in X$ we have $x \sim -1 - w - \cdots - w^{l-1}$. Thus the permutative representation of the Cuntz algebra O_∞ is an irreducible representation.

Example 3.7 (for infinite irreducible decomposition). If we take a digit set $D = \{zw + z : z \in \mathbb{Z} - \{0\}\} \cup \{w\}$, then $D_m - 1 = D_M - 1 = 0$ and $Ce(D) = \mathbb{Z} - \{0\}$. Therefore we have $F_\infty = \mathbb{Z} - \{0\}$. Since $D_m - 1 = D_M - 1 = 0$, for every $x \in X$ there exists exactly one $z \in \mathbb{Z}$ satisfying $x = z$ modulo w. For nonzero integers $z \in \mathbb{Z}$, $d_z \in D$ satisfying $z = d_z$ modulo w is $d_x = zw + z$. We now check action σ on the set $\mathbb{Z}$, which is

$$\sigma(z) = \frac{z - (zw + z)}{w} = -z$$

and

$$\sigma(-z) = \frac{-z - (zw - z)}{w} = z.$$

Thus the sequences $\{R^l(x)\}_{l=0}^\infty$ is eventually in the set $\{-z_x, z_x\}$ for some $z_x \in \mathbb{Z}_+$ for every $x \in X$. Hence for every $x \in X$ there is a positive integer $z_x \in \mathbb{Z}_+$ satisfying $x \sim z_x$, thus the permutative representation coming (2.9) is decomposed into infinitely many, as many as the cardinality of the set $\mathbb{Z}_+$, irreducible subrepresentations.

4. Realizations of the representations on $L^2(\Omega, \mu)$ and concluding remarks

The permutative representation of the Cuntz algebra O_N, N for finite, can be considered to be realized on Hilbert space $L^2(\Omega, \mu)$, where Ω is a measure space and μ is a probability measure on Ω. The representations are defined in terms of injective maps $\sigma_i : \Omega \mapsto \Omega$ with the properties $\mu(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for all $i \neq j$, and if $\rho_i = \mu(\sigma_i(\Omega))$ then $\rho_i > 0$ and $\sum_{i=1}^{N} \rho_i = 1$. In this case the set $\{\sigma_1(\Omega)\sigma_{,2}(\Omega), \cdots, \sigma_N(\Omega)\}$ is a partition of Ω up to measure zero. One such measure space Ω for N finite is a fractal set

$$T = \left\{ \sum_{i=1}^{\infty} d_{k_i} N^{-i} : d_{k_i} \in D \right\}$$

with $\rho_i = \frac{1}{N}$, where D is a residue set modulo N in $\mathbb{Z}$. This fractal sets T were studied in various papers [1], [3], [7] and [9] to study multiresolution analysis from wavelet basis. Furthermore the sets $\sigma_{i1}\sigma_{i2}\cdots\sigma_{ik}(\Omega)$ generate the measurable sets in cases of N for finite. The question is "Can we have this realization with a permutative representation of the Cuntz algebra O_∞ on Hilbert space $L^2(T_\infty)$?", where T_∞ is a subset of $\mathbb{R}$ defined by

$$\left\{ \sum_{i=1}^{\infty} d_{k_i} w^{-i} : d_{k_i} \in D,\ D \text{ is a residue set modulo } w \text{ in } X \right\}$$

The answer is negative if we have the condition $\mu(\sigma_0(T_\infty)) = \mu(\sigma_i(T_\infty))$ which condition is naturally coming from the translation invariant property of the Lebesgue measure in $\mathbb{R}$, for all $i \in \mathbb{Z}$. In fact, if we take $\delta := \mu(\sigma_0(T_\infty))$, the countable sum of δ should be the same as the measure of the set T_∞. Since Lebesgue measure of the set T_∞ is finite with the property T_∞ compact in $\mathbb{R}$, δ should be an infinitesimal. Of course it is still open if we do not apply the conditions $\Omega = T_\infty$ or $\mu(\sigma_0(T_\infty)) = \mu(\sigma_i(T_\infty))$ for all $i \in \mathbb{Z}$. One of the cases we can try is with a sequence $\{\rho_i\}, i \in \mathbb{Z}$ with

$$\rho_0 := \frac{1}{2}, \rho_i := 2^{-2i-1} \text{ for } i \in \mathbb{Z}_+ \text{ and } \rho_i := 2^{-2i} \text{ for } i \in \mathbb{Z}_-$$

for further study.

We here present a few examples of decompositions of the permutative representations.

EXAMPLE 4.1. Let the digit set $D = \{w^3 + z : z = 0, 1, 2, \cdots\} \cup \{2w^3 + z : z = -1, -2, \cdots\}$. Since $D_m - 1 = D_M - 1 = 2$, we only need to consider elements of the form $x = a_0 + a_1 w + a_2 w^2$. Then we have $Ce(D) = \{1, 2\}$ and

$$R(x) = a_1 + a_2 w - b_1 w^2, \ \ b_1 \in \{1, 2\},$$

$$R^2(x) = a_2 - b_1 w - b_2 w^2, \ b_1, b_2 \in \{1, 2\},$$

and

$$R^3(x) = -b_1 - b_2 w - b_3 w^2, \ b_1, b_2, b_3 \in \{1, 2\},$$

Since $-b_i, i \in \{1, 2, 3\}$ are negative and $x \sim 2w^3 + z$ for all negative x, we have $R^n(x) = -2 - 2w - 2w^2$ for all $6 \leqq n$ for every $x \in X$. Therefore for any $y \in X$ the sequence $\{R^l(x)\}_{l=0}^{\infty}$ converges to $-2-2w-2w^2$. Thus corresponding permutative representation is an irreducible representation even though $1 < D_m - 1 = D_M - 1$ and the set $Ce(D)$ is not a singleton set.

This example shows that the set $Ce(d)$ is a single is not a sufficient condition for irreducibility of the permutative representation.

EXAMPLE 4.2. Let $D = \{w^{|z|} + z : z \in \mathbb{Z}\}$. With a long calculation, we get either $x \sim -1$ or $x \sim -2$ or $x \sim w$ for $x \in X$. Therefore a corresponding permutative representation is decomposed into three irreducible subrepresentations even though D_M is not finite.

As we showed in the examples, the necessary and sufficient conditions on the digit set D for irreducible representation, finite decomposable representations or infinitely many decomposable representations are unknown.

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