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ISOMORHPHISMS OF (4k − 1)-DIAGONAL ALGEBRA $Alg\mathcal{L}_\infty^{(4k-1)}$

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ABSTRACT. In this paper, we introduce the $(4k-1)$-diagonal algebra $Alg\mathcal{L}_\infty^{(4k-1)}$ and investigate the necessary and sufficient condition that isomorphisms of $Alg\mathcal{L}_\infty^{(4k-1)}$ are quasi-spatial.

1. Introduction

The study of reflexive, but not necessary self-adjoint, algebras of Hilbert space operators has become one of the fastest growing specialties in operator theory [4,5,6]. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. In [2], the 7-diagonal algebra $Alg\mathcal{L}_\infty^{(7)}$ which is reflexive algebra, is introduced. And we investigated the necessary and sufficient condition that isomorphisms of $Alg\mathcal{L}_\infty^{(7)}$ are quasi-spatial. In this paper we study isomorphisms of a certain reflexive algebra $Alg\mathcal{L}_\infty^{(4k-1)}$ which is a generalization of $Alg\mathcal{L}_\infty^{(7)}$.

First we introduce the terminologies used in this paper. Let $\mathcal{H}$ be a complex separable Hilbert space. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$, containing 0 and I. If $\mathcal{L}$ is a subspace lattice, $Alg\mathcal{L}$ denotes the algebra of all bounded operators on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. $Alg\mathcal{L}$ is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $Lat\mathcal{A}$ is the lattice of all projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if

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$\mathcal{A} = Alg Lat \mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = Lat Alg \mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is then called a CSL-algebra. An algebra $\mathcal{A}$ is $(4k-1)$-diagonal if there exists a countable partition $\{E_i\}$ of $\mathcal{H}$ so that every $A \in \mathcal{A}$ is block $(4k-1)$-diagonal with respect to the sequence $E_1, E_2, \cdots$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is said to be spatially implemented, or simply spatial, if there is a bounded invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_1$. A slightly weaker condition is that φ be quasi-spatial; in this case we drop the assumption that T be bounded but we require that T be one-to-one with dense domain $\mathcal{D}$, that $\mathcal{D}$ be an invariant linear manifold for $Alg\mathcal{L}_1$, and that

$$\varphi(A)Tf = TAf$$

for all A in $Alg\mathcal{L}_1$ and $f \in \mathcal{D}$. Any quasi-spatial isomorphism is automatically continuous in the norm topology [7]. If $x_1, x_2, \cdots, x_m$ are vectors in some Hilbert space, we denote by $[x_1, x_2, \cdots, x_m]$ the closed subspace spanned by the vectors $x_1, x_2, \cdots, x_m$.

Let $\mathcal{H}$ be infinite dimensional separable complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots\}$ and let k be some natural number. Let $\mathcal{L}_\infty^{(4k-1)}$ be the subspace lattice of orthogonal projections generated by $\{[e_{2i-1}], [e_{2i}, e_{2i-2k+1}, e_{2i-2k+3}, \cdots, e_{2i+2k-1}] : i = 1, 2, \cdots\}$, where $e_p = 0$ if $p \leq 0$. Then $Alg\mathcal{L}_\infty^{(4k-1)}$ is a $(4k-1)$-diagonal algebra. If $k = 1$, then the algebra $Alg\mathcal{L}_\infty^{(3)}$ is tridiagonal, which was introduced by F. Gilfeather and D. Larson [5], and isomorphisms of this algebra are quasi-spatial [7]. If $k \geq 2$, then isomorphisms of $Alg\mathcal{L}_\infty^{(4k-1)}$ need not be quasi-spatial [2]. In this paper we will investigate the necessary and sufficient condition that isomorphisms of $Alg\mathcal{L}_\infty^{(4k-1)}$ are quasi-spatial.

2. Isomorphisms of $Alg\mathcal{L}_\infty^{(4k-1)}$

Subspace lattices $\mathcal{L}$ need not be reflexive however, commutative subspace lattices are reflexive [1]. Since the lattice $\mathcal{L}_\infty^{(4k-1)}$ is commutative, it is reflexive CSL. If we put $Alg\mathcal{L}_{\infty}^{(4k-1)} = \mathcal{A}$, then $LatAlg\mathcal{L}_{\infty}^{(4k-1)} = \mathcal{L}_{\infty}^{(4k-1)}$ and so $Lat\mathcal{A} = \mathcal{L}_{\infty}^{(4k-1)}$. From this we have the following theorem.

LEMMA 2.1. *An algebra $Alg\mathcal{L}_{\infty}^{(4k-1)}$ is non-self-adjoint reflexive CSL-algebra.*

Let i and j be two nonzero natural numbers. Then E_{ij} is the matrix whose (i,j)-component is 1 and all other entries are zero. If $\rho : Alg\mathcal{L}_{\infty}^{(4k-1)} \to Alg\mathcal{L}_{\infty}^{(4k-1)}$ is an isomorphism such that $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$. then $\rho(E_{ij}) = \rho(E_{ii}E_{ij}E_{jj}) = E_{ii}\rho(E_{ij})E_{jj}$. From this we have the following theorem.

THEOREM 2.2. *Let $\rho : Alg\mathcal{L}_{\infty}^{(4k-1)} \to Alg\mathcal{L}_{\infty}^{(4k-1)}$ be an isomorphism such that $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$. Then there exist nonzero complex numbers γ_{ij} such that $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_{\infty}^{(4k-1)}$.*

THEOREM 2.3. *Let $\rho : Alg\mathcal{L}_{\infty}^{(4k-1)} \to Alg\mathcal{L}_{\infty}^{(4k-1)}$ be an isomorphism such that $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$ and let $\rho(E_{ij}) = \gamma_{ij}E_{ij}$, $\gamma_{ij} \neq 0$, for all E_{ij} in $Alg\mathcal{L}_{\infty}^{(4k-1)}$. If $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{\infty}^{(4k-1)}$, then there is a diagonal (possibly unbounded) operator T defined on the (non-closed) linear span $\mathcal{D}$ of the basis vectors such that $\rho(A)Tx = TAx$ for all $x \in \mathcal{D}$ and $A \in Alg\mathcal{L}_{\infty}^{(4k-1)}$.*

PROOF. Let $T = diag(t_1, t_2, \cdots)$, where

$$t_1 = 1,$$

$$t_2 = \gamma_{12}^{-1},$$

$$t_{2i-1,2i-1} = \prod_{j=1}^{i-1} \gamma_{2j+1,2j} \Big(\prod_{j=1}^{i-1} \gamma_{2j-1,2j}\Big)^{-1},$$

$$t_{2i,2i} = \prod_{j=1}^{i-1} \gamma_{2j+1,2j} \Big(\prod_{j=1}^{i} \gamma_{2j-1,2j}\Big)^{-1}$$

for all $i = 1, 2, \cdots$. Let $A = [a_{ij}]$ be in $Alg\mathcal{L}_{\infty}^{(4k-1)}$. Then comparing the components of $\rho(A)T$ with those of TA we have $\rho(A)T = TA$. □

THEOREM 2.4. *Let ρ be as in* Theorem 2.3. *If ρ is quasi-spatially implemented by some one-to-one operator T with dense domain $\mathcal{D}$, then T is diagonal and $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all pairs $(p,q),(s,t),(p,t)$ and (s,q) with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_\infty^{(4k-1)}$.*

PROOF. Let $T = [t_{ij}]$ be a one-to-one operator with dense domain $\mathcal{D}$ such that $\rho(A)Tx = TAx$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$ and $x \in \mathcal{D}$. Since $\rho(E_{ii})T = TE_{ii}$ and $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$, we have $E_{ii}T = TE_{ii}$. So $t_{ij} = 0$ for all $i, j (i \neq j)$ and $t_{ii} \neq 0$ for all $i = 1, 2, \cdots$. Hence T is diagonal. Let $T = \sum_{i=1}^{\infty} t_{ii}E_{ii}$ and $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$. Then for any $p, q (p \neq q)$ with E_{pq} in $Alg\mathcal{L}_\infty^{(4k-1)}$,

$$\rho(E_{pq})T = \gamma_{pq}E_{pq}\left(\sum_{i=1}^{\infty} t_{ii}E_{ii}\right) = \gamma_{pq}t_{qq}E_{pq}$$

and

$$TE_{pq} = \left(\sum_{i=1}^{\infty} t_{ii}E_{ii}\right)E_{pq} = t_{pp}E_{pq}.$$

Since $\rho(E_{pq})T = TE_{pq}$, we have $\gamma_{pq}t_{qq} = t_{pp}$. Hence $\gamma_{pq} = t_{pp}t_{qq}^{-1}$. If E_{st} is in $Alg\mathcal{L}_\infty^{(4k-1)}$, then $\gamma_{st} = t_{ss}t_{tt}^{-1}$. Hence $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_\infty^{(4k-1)}$. □

LEMMA 2.5 [5]. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$,respectively, and suppose that $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is an algebraic isomorphism. Let $\mathcal{M}$ be maximal abelian self-adjoint subalgebra (masa) contained in $Alg\mathcal{L}_1$. Then there exists a bounded invertible operator $Y : \mathcal{H}_1 \to \mathcal{H}_2$ and an automorphism $\rho : Alg\mathcal{L}_1 \to Alg\mathcal{L}_1$ such that*

(i) $\rho(M) = M$ *for all M in $\mathcal{M}$ and*
(ii) $\varphi(A) = Y\rho(A)Y^{-1}$ *for all A in $Alg\mathcal{L}_1$.*

LEMMA 2.6. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism. Then there exists a bounded invertible operator Y in $\mathcal{B}(\mathcal{H})$ and an isomorphism $\rho : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ satisfying $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$ such that $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$.*

PROOF. Let $\mathcal{M} = (Alg\mathcal{L}_\infty^{(4k-1)}) \bigcap (Alg\mathcal{L}_\infty^{(4k-1)})^*$. Then $\mathcal{M}$ is a masa of $Alg\mathcal{L}_\infty^{(4k-1)}$ and E_{ii} is in $\mathcal{M}$ for all $i = 1, 2, \cdots$. By Lemma 2.5, there exist a bounded invertible operator Y in $\mathcal{B}(\mathcal{H})$ and an automorphism $\rho : Alg\mathcal{L}_\infty^{(4k-1)} \rightarrow Alg\mathcal{L}_\infty^{(4k-1)}$ satisfying $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$ such that $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$. □

LEMMA 2.7. *Let φ, ρ and Y be as in* Lemma 2.6 *and let $Y = [y_{ij}]$. Then we have the following.*

(1) *$y_{2i,2j-1} = 0$ for all $i, j = 1, 2, \cdots$.*
(2) *If $y_{2i,2j} \neq 0$, then $y_{2k,2l} = 0$ for all positive integers k and l such that $k \neq i, l \neq j$.*
(3) *If $y_{2i-1,2j-1} \neq 0$, then $y_{2k-1,2l-1} = 0$ for all positive integers k and l such that $k \neq i, l \neq j$.*

PROOF. Let $\varphi(A) = [b_{ij}]$ and $\rho(A) = [a_{ij}]$ be in $Alg\mathcal{L}_\infty^{(4k-1)}$ and let $Y = [y_{ij}]$. Then from Lemma 2.6, $\varphi(A)Y = Y\rho(A)$. Comparing the components of $\varphi(A)T$ with that of $T\rho(A)$, we have above result. □

From Lemma 2.7, we have the following lemma.

LEMMA 2.8. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \rightarrow Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism. Then for all $j = 1, 2, \cdots$,*

$$\varphi(E_{2j-1,2j-1}) = E_{2p-1,2p-1} + \textstyle\sum_l \alpha_{2p-1,2l} E_{2p-1,2l}$$
$$\varphi(E_{2j,2j}) = E_{2q,2q} + \textstyle\sum_m \alpha_{2m-1,2q} E_{2m-1,2q}$$

for some $p, q = 1, 2, \cdots$ and some complex numbers $\alpha_{2p-1,2l}$ $\alpha_{2m-1,2q}$ with $E_{2p-1,2l}$, $E_{2m-1,2q}$ in $Alg\mathcal{L}_\infty^{(4k-1)}$.

With the similar proof as Theorem 3.6 and Theorem 3.8 in [4], we can get the following theorem.

LEMMA 2.9. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \rightarrow Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism. Then there is nonzero complex numbers β_{ij} and complex numbers α_{ij} for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$ such that*

$\varphi(E_{2p-1,2p-1}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j} E_{2p-1,2j}$,
$\varphi(E_{2q,2q}) = E_{2q,2q} - \sum_j \alpha_{2j-1,2q} E_{2j-1,2q}$ *and*
$\varphi(E_{2p-1,2q}) = \beta_{2p-1,2q} E_{2p-1,2q}$ *for all* $E_{2p-1,2q}$ *in* $Alg\mathcal{L}_\infty^{(4k-1)}$.

LEMMA 2.10. *Let φ, ρ and Y be as in* Lemma 2.6. *Then Y is in $Alg\mathcal{L}_\infty^{(4k-1)}$.*

PROOF. Since $\varphi(E_{2i-1,2i-1})Y = Y\rho(E_{2i-1,2i-1})$, we have

$$\left(E_{2i-1,2i-1} + \sum_q \alpha_{2i-1,2q}\right)[y_{ij}] = [y_{ij}]E_{2i-1,2i-1}.$$

Hence $y_{m,2i-1} = 0$ for all m with $m \neq 2i-1$. Since $\varphi(E_{2j,2j})Y = Y\rho(E_{2j,2j})$, we have

$$\left(E_{2j,2j} + \sum_p \alpha_{2p-1,2j}\right)[y_{ij}] = [y_{ij}]E_{2j,2j}.$$

Hence $y_{2j,m} = 0$ for all m with $m \neq 2j$ and $y_{2k+2j+m,2j} = 0$ for all $m, j = 1, 2, \cdots$. Since $\varphi(E_{11})Y = Y\rho(E_{11})$ and $y_{2l,2k+2t} = 0$ for all $l = 1, 2, \cdots, k$ and $t = 1, 2, \cdots$, we have $y_{1,2k+2t} = 0$ for all $t = 1, 2, \cdots$. Since $\varphi(E_{33})Y = Y\rho(E_{33})$ and $y_{2l+2,2k+2t} = 0$ for all $l = 1, 2, \cdots, k$ and $t = 2, 3, \cdots$, we have $y_{3,2k+2t} = 0$ for all $t = 2, 3, \cdots$. Similarly $y_{2i-1,2k+2t} = 0$ for $t = i, i+1, \cdots$. Hence Y is in $Alg\mathcal{L}_\infty^{(4k-1)}$. □

If we summarize Lemmas, then we can get the following theorem.

THEOREM 2.11. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism. Then there is a bounded invertible operator Y in $Alg\mathcal{L}_\infty^{(4k-1)}$ and an isomorphism $\rho : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ satisfying $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$ such that $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$.*

From Theorem 2.4 and Theorem 2.11, we have the following theorems.

THEOREM 2.12. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism. If φ is quasi-spatially implemented by T, then T is an one-to-one operator of the form $T = \sum t_{ij}E_{ij}$ for all i, j with E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$.*

LEMMA 2.13. *Let φ, ρ and $Y = [y_{ij}]$ be as in* Theorem 2.11. *Let $\rho(E_{2p-1,2q}) = \gamma_{2p-1,2q}E_{2p-1,2q}$ and $\varphi(E_{2p-1,2q}) = \beta_{2p-1,2q}E_{2p-1,2q}$ for all $E_{2p-1,2q}$ in $Alg\mathcal{L}_\infty^{(4k-1)}$. Then $\beta_{2p-1,2q} = y_{2q,2q}^{-1}y_{2p-1,2p-1}\gamma_{2p-1,2q}$.*

PROOF. Let $E_{2p-1,2q} \in Alg\mathcal{L}_\infty^{(4k-1)}$. Comparing the $(2p-1,2q)$-component of $\varphi(E_{2p-1,2q})Y$ with that of $Y\rho(E_{2p-1,2q})$, we have

$$\beta_{2p-1,2q}y_{2q,2q} = \gamma_{2p-1,2q}y_{2p-1,2p-1}.$$

Hence

$$\beta_{2p-1,2q} = y_{2q,2q}^{-1}y_{2p-1,2p-1}\gamma_{2p-1,2q}.$$

□

LEMMA 2.14. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism and $\rho : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ an isomorphism satisfying $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$ such that $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$. Then ρ is quasi-spatial if and only if φ is quasi-spatial.*

PROOF. Suppose that ρ is quasi-spatial. Then there is an invertible operator T defined on the linear span $\mathcal{D}$ of the basis vector such that $\rho(A)Tx = TAx$ for all x in $\mathcal{D}$ and A in $Alg\mathcal{L}_\infty^{(4k-1)}$. Since $Y^{-1}\varphi(A)Y = \rho(A)$, we have $Y^{-1}\varphi(A)YTx = TAx$ for all x in $\mathcal{D}$ and A in $Alg\mathcal{L}_\infty^{(4k-1)}$. Hence $\varphi(A)YTx = YTAx$ for all x in $\mathcal{D}$ and A in $Alg\mathcal{L}_\infty^{(4k-1)}$. □

THEOREM 2.15. *Let $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ be an isomorphism and $\varphi(E_{ij}) = \beta_{ij}E_{ij}$ for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$. Then $\beta_{pq}\beta_{st} = \beta_{pt}\beta_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_\infty^{(4k-1)}$ if and only if φ is quasi-spatial.*

PROOF. Suppose that $\rho : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ is an isomorphism such that $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$ and $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots$. Then there exist nonzero complex numbers γ_{ij} such that $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$. By Lemma 2.13, $\beta_{2p-1,2q} = y_{2q,2q}^{-1}y_{2p-1,2p-1}\gamma_{2p-1,2q}$, where $Y = [y_{st}]$. Hence $\beta_{pq}\beta_{st} = \beta_{pt}\beta_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_\infty^{(4k-1)}$ if and only if $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_\infty^{(4k-1)}$ if and only if ρ is quasi-spatial if and only if φ is quasi-spatial. □

THEOREM 2.16. *Let T be an invertible operator of the form $T = \sum t_{ij}E_{ij}$ for all i, j with E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$. Then $TAT^{-1} \in Alg\mathcal{L}_\infty^{(4k-1)}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$ if and only if*

$$\sup\left\{\left|\frac{t_{2n-2i+1,2n-2i+1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n-2i+1,2n}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1 \le i \le k, n = 1, 2, \cdots\right\} < \infty.$$

PROOF. Suppose that $TAT^{-1} \in Alg\mathcal{L}_\infty^{(4k-1)}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$. Let A be the matrix whose $(2n-1, 2n+2i-2)$-component is 1 for all positive integers n and all $i(1 \le i \le k)$ and all other entries are 0. Then $A \in Alg\mathcal{L}_\infty^{(4k-1)}$ and so $TAT^{-1} \in Alg\mathcal{L}_\infty^{(4k-1)}$. Now the $(2n-1, 2n+2i-2)$-component of TAT^{-1} is $\frac{t_{2n-1,2n-1}}{t_{2n+2i-2,2n+2i-2}}$ for all positive integers n and all $i(1 \le i \le k)$. Hence

$$\sup\left\{\left|\frac{t_{2n-1,2n-1}}{t_{2n+2i-2,2n+2i-2}}\right| : n = 1, 2, \cdots \text{ and } 1 \le i \le k\right\} < \infty.$$

Let B be the matrix whose $(2n+2i-1, 2n)$-component is 1 for all positive integers n and all $i(1 \le i \le k)$ and all other entries are 0. In the same way we can show that

$$\sup\left\{\left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right| : n = 1, 2, \cdots \text{ and } 1 \le i \le k\right\} < \infty.$$

Let C be the diagonal operator whose $(2n-1, 2n-1)$-component is 1 and $(2n, 2n)$-component is 2 for all positive integers n. Then $C \in Alg\mathcal{L}_\infty^{(4k-1)}$ and so $TCT^{-1} \in Alg\mathcal{L}_\infty^{(4k-1)}$. Since TCT^{-1} is the matrix whose

(1) $(2n, 2n)$-component is 2
(2) $(2n-1, 2n-1)$-component is 1
(3) $(2n-1, 2n+2i-2)$-component is $\frac{t_{2n-1,2n+2i-2}}{t_{2n+2i-2,2n+2i-2}}$ for $1 \le i \le k$
(4) $(2n+2i-1, 2n)$-component is $\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}$ for $1 \le i \le k$
(5) all other entries are 0 for all positive integers n,

we have

$$\sup\left\{\left|\frac{t_{2n-1,2n+2i-2}}{t_{2n+2i-2,2n+2i-2}}\right|,\left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1\le i\le k, n=1,2,\cdots\right\}<\infty.$$

Thus

$$\sup\left\{\left|\frac{t_{2n-2i+1,2n-2i+1}}{t_{2n,2n}}\right|,\left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right|,\left|\frac{t_{2n-2i+1,2n}}{t_{2n,2n}}\right|,\left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1\le i\le k, n=1,2,\cdots\right\}<\infty.$$

Conversely suppose that

$$\sup\left\{\left|\frac{t_{2n-2i+1,2n-2i+1}}{t_{2n,2n}}\right|,\left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right|,\left|\frac{t_{2n-2i+1,2n}}{t_{2n,2n}}\right|,\left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1\le i\le k, n=1,2,\cdots\right\}<\infty.$$

Let $A=[a_{ij}]\in Alg\mathcal{L}_\infty^{(4k-1)}$. Then TAT^{-1} has the matrix representation whose

(1) (n,n)-component is a_{nn}

(2) $(2n-1,2n+2i-2)$-component is

$$t^{-1}_{2n+2i-2,2n+2i-2}(t_{2n-1,2n+2i-2}(a_{2n+2i-2,2n+2i-2}-a_{2n-1,2n-1}) + t_{2n-1,2n-1}a_{2n-1,2n+2i-2})$$

(3) $(2n+2i-1,2n)$-component is

$$t^{-1}_{2n,2n}(t_{2n+2i-1,2n}(a_{2n,2n}-a_{2n+2i-1,2n+2i-1}) + t_{2n+2i-1,2n+2i-1}a_{2n+2i-1,2n})$$

(4) all other entries are 0

for all positive integers n and $1\le i\le k$.

Let B_0 be the diagonal operator whose (n,n)-component is a_{nn} for all positive integers n. Let $B_{1,i}$ be the matrix whose $(2n-1,2n+2i-2)$-component is

$$\frac{t_{2n-1,2n+2i-2}(a_{2n+2i-2,2n+2i-2}-a_{2n-1,2n-1})}{t_{2n+2i-2,2n+2i-2}}$$

for all positive integers n and $1 \leq i \leq k$ and all other entries are 0 and let $B'_{1,i}$ be the matrix whose $(2n-1, 2n+2i-2)$-component is

$$\frac{t_{2n-1,2n-1}a_{2n-1,2n+2i-2}}{t_{2n+2i-2,2n+2i-2}}$$

for all positive integers n and $1 \leq i \leq k$ and all other entries are 0. Let $B_{2,i}$ be the matrix whose $(2n+2i-1, 2n)$-component is

$$\frac{t_{2n+2i-1,2n}(a_{2n,2n} - a_{2n+2i-1,2n+2i-1})}{t_{2n,2n}}$$

for all positive integers n and $1 \leq i \leq k$ and all other entries are 0 and let $B'_{2,i}$ be the matrix whose $(2n+2i-1, 2n)$-component is

$$\frac{t_{2n+2i-1,2n+2i-1}a_{2n+2i-1,2n}}{t_{2n,2n}}$$

for all positive integers n and $1 \leq i \leq k$ and all other entries are 0. Then $TAT^{-1} = B_0 + \sum_{i=1}^{k}(B_{1,i} + B'_{1,i} + B_{2,i} + B'_{2,i})$. By the hypothesis,

$$\sup\left\{\left|\frac{t_{2n-1,2n+2i-2}}{t_{2n+2i-2,2n+2i-2}}\right|, \left|\frac{t_{2n-1,2n-1}}{t_{2n+2i-2,2n+2i-2}}\right|, \left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right|, \right.$$
$$\left.\left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right| : 1 \leq i \leq k, n = 1, 2, \cdots\right\} < \infty.$$

Since

$$\sup\{|a_{2n+2i-2,2n+2i-2} - a_{2n-1,2n-1}|, |a_{2n,2n} - a_{2n+2i-1,2n+2i-1}|$$
$$: 1 \leq i \leq k, n = 1, 2, \cdots\} < \infty,$$

$B_0, B_{1,i}, B'_{1,i}, B_{2,i}$ and $B'_{2,i}$ belong to $Alg\mathcal{L}_\infty^{(4k-1)}$ for all $i (1 \leq i \leq k)$. Thus TAT^{-1} belongs to $Alg\mathcal{L}_\infty^{(4k-1)}$. □

THEOREM 2.17. *A map $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ is an isomorphism such that φ is implemented by a (possibly unbounded) T, that is $\varphi(A) = TAT^{-1}$ if and only if*

$$\sup\left\{\left|\frac{t_{2n-2i+1,2n-2i+1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n-2i+1,2n}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1 \le i \le k, n = 1,2,\cdots\right\} < \infty.$$

PROOF. Suppose $\varphi(A) = TAT^{-1}$ for some invertible operator (not necessary bounded) $T = [t_{ij}]$. By Theorem 2.12, $T = [t_{ij}]$ is a matrix of the form $T = \sum t_{ij}E_{ij}$ for all i,j with E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$. From Theorem 2.16,

$$\sup\left\{\left|\frac{t_{2n-2i+1,2n-2i+1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n-2i+1,2n}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1 \le i \le k, n = 1,2,\cdots\right\} < \infty.$$

Conversely, suppose that $T = \sum t_{ij}E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_\infty^{(4k-1)}$ satisfying

$$\sup\left\{\left|\frac{t_{2n-2i+1,2n-2i+1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n+2i-1}}{t_{2n,2n}}\right|, \left|\frac{t_{2n-2i+1,2n}}{t_{2n,2n}}\right|, \left|\frac{t_{2n+2i-1,2n}}{t_{2n,2n}}\right| : 1 \le i \le k, n = 1,2,\cdots\right\} < \infty.$$

Define $\varphi : Alg\mathcal{L}_\infty^{(4k-1)} \to Alg\mathcal{L}_\infty^{(4k-1)}$ by $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$. Then TAT^{-1} is in $Alg\mathcal{L}_\infty^{(4k-1)}$ for all A in $Alg\mathcal{L}_\infty^{(4k-1)}$ and so φ is well define. It is clear that φ is an isomorphism. □

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