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ESTIMATES OF CHRISTOFFEL FUNCTIONS FOR GENERALIZED POLYNOMIALS WITH EXPONENTIAL WEIGHTS

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ABSTRACT. Generalized nonnegative polynomials are defined as the products of nonnegative polynomials raised to positive real powers. The generalized degree can be defined in a natural way. We extend some results on Infinite-Finite range inequalities, Christoffel functions, and Nikolskiĭ type inequalities corresponding to weights $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, to those for generalized nonnegative polynomials.

1. Introduction and notation

We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, the set of positive integers, the set of real numbers, and the set of complex numbers, respectively. $\mathbb{R}^+$ denotes the set of positive real numbers.

We denote by $\mathbb{P}_n$ $(n \in \mathbb{N})$, the set of all real algebraic polynomials of degree at most n.

The function

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j}$$

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with $r_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$, and $0 \neq \omega \in \mathbb{C}$ is called a generalized nonnegative algebraic polynomial of generalized degree

$$n \stackrel{\text{def}}{=} \sum_{j=1}^{m} r_j \, .$$

We denote by GANP_n the set of all generalized nonnegative algebraic polynomials of degree at most $n \in \mathbb{R}^+$.

Note that, here, $n > 0$ is not necessarily an integer. In fact, we assume throughout this paper that $n \in \mathbb{R}^+$ unless stated otherwise.

In this paper we study generalized nonnegative polynomials restricted to the real line. Using

$$|z - z_j|^{r_j} = ((z - z_j)(z - \bar{z}_j))^{r_j/2} \, , \qquad z \in \mathbb{R},$$

we can easily check that when $f \in \mathrm{GANP}_n$ is restricted to the real line, then it can be written as

$$f = \prod_{j=1}^{m} P_j^{r_j/2} \, , \quad 0 \leq P_j \in \mathbb{P}_2 \, , \quad r_j \in \mathbb{R}^+, \quad \sum_{j=1}^{m} r_j \leq n \, ,$$

which is the product of nonnegative polynomials raised to positive real powers. This explains the name *generalized nonnegative polynomials.* Observe that if $f \in \mathrm{GANP}_n$ with $r_j \geq 1$ in its representation then one-sided derivatives of f exist for all $x \in \mathbb{R}$ with the same absolute value, thus, $|f'(x)|$ is well defined for all $x \in \mathbb{R}$. Many properties of generalized nonnegative polynomials were investigated in a series of papers (cf. [1–4]).

In what follows we denote by $a_n = a_n(\alpha)$ the *Mhaskar-Rahmanov-Saff* number which is the positive solution of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t)(1 - t^2)^{-\frac{1}{2}} dt \, , \qquad n \in \mathbb{R}^+,$$

where $Q(x) = |x|^\alpha$, $\alpha > 0$. Explicitly,

$$a_n = a_n(\alpha) = \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha} , \quad n \in \mathbb{R}^+,$$

where

$$\lambda_\alpha = \frac{2^{2-\alpha}\Gamma(\alpha)}{\{\Gamma(\alpha/2)\}^2} \, .$$

Now let $0 < p < \infty$. Then the generalized Christoffel function for ordinary polynomials is defined by

$$\lambda_{n,p}(W_\alpha; x) = \min_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|P(t)W_\alpha(t)|^p}{|P(x)|^p} dt\,, \quad x \in \mathbb{R}\,,\ n \in \mathbb{N}\,.$$

The generalized Christoffel function for generalized nonnegative polynomials is defined by[1]

$$\omega_{n,p}(W_\alpha; x) = \inf_{f \in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{(f(t)W_\alpha(t))^p}{f^p(x)} dt\,, \quad x \in \mathbb{R}\,,\ n \in \mathbb{R}^+.$$

We write $g_n(x) \sim h_n(x)$ if for every n and for every x in consideration

$$0 < c_1 \leq \frac{g_n(x)}{h_n(x)} \leq c_2 < \infty,$$

and $g(x) \sim h(x)$, $n \sim N$ have similar meanings.

We denote by $m(\Delta)$ one-dimensional Lebesgue measure of a set $\Delta \subset \mathbb{R}$.

Finally, for each $n \in \mathbb{R}^+$, the symbol $[n]$ denotes the integer part of n.

There are many inequalities in conjunction with the weight $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, (cf. [5–12]). The majority of these inequalities, which hold for ordinary polynomials, are expected to be true for generalized nonnegative polynomials. In this paper we extend some of these inequalities to those for generalized nonnegative polynomials. For example, Mhaskar and Saff [12, Theorem 2.7, p. 210] proved that

$$\| PW_\alpha \|_{L^\infty(\mathbb{R})} = \| PW_\alpha \|_{L^\infty([-a_n, a_n])}\,, \quad P \in \mathbb{P}_n\,,\ n \in \mathbb{N}\,.$$

As an analogue, we show that

$$\| fW_\alpha \|_{L^\infty(\mathbb{R})} = \| fW_\alpha \|_{L^\infty([-a_n, a_n])}\,, \quad f \in \mathrm{GANP}_n\,,\ n \in \mathbb{R}^+.$$

We also find lower and upper bounds of Christoffel functions for generalized polynomials.

The rest of this paper is organized as follows. In Section 2, we state our results. In Section 3, we give the proof of theorems.

2. Results

In this section we state the results which will be proved in Section 3.

[1] ω=omega.

2.1. Infinite-Finite range inequalities

In the theory of orthogonal polynomials for weights on the whole real line, *Infinite-Finite range inequalities* are useful because they reduce problems over an infinite interval to problems on a finite interval. For ordinary polynomials, Mhaskar and Saff [12, Theorem 2.7, p. 210] established, in an asymptotic sense, best possible inequalities. They showed that

$$\| PW_\alpha \|_{L^\infty(\mathbb{R})} = \| PW_\alpha \|_{L^\infty([-a_n,a_n])}\,, \quad P \in \mathbb{P}_n\,, \quad n \in \mathbb{N}\,.$$

Having read a preliminary draft of this paper, D. S. Lubinsky suggested that we could also use the method of Mhaskar and Saff, to prove Infinite-Finite range inequalities for generalized nonnegative polynomials. Using this idea, we were able to extend Infinite-Finite range inequalities for ordinary polynomials to those for generalized nonnegative polynomials as follows.

THEOREM 2.1. *Let $\epsilon > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$. If $p = \infty$, then*

$$\| fW_\alpha \|_{L^\infty(\mathbb{R})} = \| fW_\alpha \|_{L^\infty([-a_n,a_n])}\,,$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.
If $0 < p < \infty$, then there exist positive constants C_1 and C_2 so that, whenever

$$\epsilon \le n \in \mathbb{R}^+, \qquad \frac{n}{(\log(n+1))^2} \ge K_n \ge C_1\,, \qquad \text{and}$$

$$\delta_n = \left(\frac{K_n \log(n+1)}{n}\right)^{2/3},$$

then

$$\| fW_\alpha \|_{L^p(\mathbb{R})} \le \left(1 + (n+1)^{-C_2K_n}\right) \| fW_\alpha \|_{L^p([-a_n(1+\delta_n),a_n(1+\delta_n)])}\,,$$

for all $f \in \mathrm{GANP}_n$.

From now on, the condition $\epsilon > 0$ appears in the statement of theorems whenever Theorem 2.1, for $0 < p < \infty$, is used.

THEOREM 2.2. *Let $\epsilon > 0$ and $d > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. Let*

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}, \quad n \in \mathbb{R}^+.$$

If $0 < p < \infty$, *then there exist positive constants* B^* *and* C_1 *such that for all measurable sets* $\Delta_n \subset [-B^* a_n, B^* a_n]$ *with* $m(\Delta_n) \leq s_n/2$,

$$\int_{-\infty}^{\infty} f^p(x) W_\alpha^p(x) dx \leq C_1 \int_{\substack{|x| \leq B^* a_n \\ x \notin \Delta_n}} f^p(x) W_\alpha^p(x) dx ,$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$.

If $p = \infty$, *then there exists a positive constant* C_2 *such that for all measurable sets* $\Delta_n \subset [-B^* a_n, B^* a_n]$ *with* $m(\Delta_n) \leq s_n$,

$$\| fW_\alpha \|_{L^\infty(\mathbb{R})} \leq C_2 \| fW_\alpha \|_{L^\infty([-B^* a_n, B^* a_n] \setminus \Delta_n)} ,$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.

2.2. Lower and upper bounds for Christoffel functions

Christoffel functions play an important role in the theory of orthogonal polynomials. For ordinary polynomials the estimates of Christoffel functions for W_α appear in, for instance, [7, Theorem 1.1, p. 465]. The following is a generalization for generalized nonnegative polynomials.

THEOREM 2.3. *Let* $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, *and* $0 < p < \infty$. *Then*

$$\omega_{n,p}(W_\alpha; x) \geq C_{\alpha,p} \frac{a_n}{n} W_\alpha^p(x) , \qquad x \in \mathbb{R}, \ n \in \mathbb{R}^+,$$

and

$$\omega_{n,p}(W_\alpha; x) \leq \lambda_{[n]+1,p}(W_\alpha; x) , \qquad x \in \mathbb{R}, \ n \in \mathbb{R}^+.$$

REMARK. If $\alpha > 1$, there exist positive constants C_1 and C_2 depending on p and α, such that

$$\lambda_{[n]+1,p}(W_\alpha; x) \leq C_1 \frac{a_n}{n} W_\alpha^p(x) , \qquad |x| \leq C_2 a_n .$$

(see, for example, [6, Theorem 7.4, p. 166]).

2.3. Nikolskiĭ type inequalities

THEOREM 2.4. *Let* $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, *and* $0 < p < r \leq \infty$. *Then there exists a positive constant* $C(\alpha, p, r)$ *such that*

$$\| fW_\alpha \|_{L^r(\mathbb{R})} \leq C(\alpha, p, r) \left(\frac{n}{a_n} \right)^{\frac{1}{p} - \frac{1}{r}} \| fW_\alpha \|_{L^p(\mathbb{R})} ,$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.

In the opposite direction we have

THEOREM 2.5. *Let $\epsilon > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, and $0 < p < r \le \infty$. Then there exists a positive constant $C(\alpha, \epsilon, p, r)$ such that*

$$\| fW_\alpha \|_{L^p(\mathbb{R})} \le C(\alpha, \epsilon, p, r)(a_n)^{\frac{1}{p}-\frac{1}{r}} \| fW_\alpha \|_{L^r(\mathbb{R})} ,$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \le n \in \mathbb{R}^+$.

For ordinary polynomials, Theorems 2.4 and 2.5 can be found in, for instance, [12, Theorems 6.1 and 6.4, pp. 228–231].

3. Proof of Theorems

To prove Theorem 2.1, we need the following lemma.

LEMMA 3.1. *Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$. Let $0 < p < \infty$. Then there exists $C > 0$ such that*

$$\| fW_\alpha \|_{L^\infty(\mathbb{R})} \le C(n+1)^{2/p} \| fW_\alpha \|_{L^p(\mathbb{R})} ,$$

for $f \in \mathrm{GANP}_n$.

PROOF OF LEMMA 3.1. Let $f \in \mathrm{GANP}_n$ and let ξ be such that

$$|f(\xi)W_\alpha(\xi)| \ge \frac{1}{2} \| fW_\alpha \|_{L^\infty(\mathbb{R})} .$$

Suppose first that $\xi \ge 1$. Then, by [3, Theorem 6, p. 246],

$$\begin{aligned} \| fW_\alpha \|_{L^p(\mathbb{R})} &\ge \| fW_\alpha \|_{L^p([\xi-1,\xi])} \\ &\ge W_\alpha(\xi) \| f \|_{L^p([\xi-1,\xi])} \\ &\ge c(n+1)^{-2/p} W_\alpha(\xi) \| f \|_{L^\infty([\xi-1,\xi])} \\ &\ge c(n+1)^{-2/p} |f(\xi)W_\alpha(\xi)| \\ &\ge \frac{c}{2}(n+1)^{-2/p} \| fW_\alpha \|_{L^\infty(\mathbb{R})} . \end{aligned}$$

If $\xi \le -1$, the result follows similarly. If $|\xi| \le 1$, the result again follows in a similar way, since $W_\alpha^{\pm}(x)$ is bounded in $[-1, 1]$. □

In proving Infinite-Finite range inequalities for ordinary polynomials, Mhaskar and Saff used the function

$$U_\alpha(z) = \int_{-1}^{1} \log|z-t| v(\alpha; t) dt - \frac{|z|^\alpha}{\lambda_\alpha} + \log 2 + \frac{1}{\alpha} ,$$

$$(\alpha > 0, \qquad z \in \mathbb{C}),$$

where $v(\alpha;t)$ is the Ullman distribution

$$v(\alpha;t)=\frac{\alpha}{\pi}\int_{|t|}^{1}y^{\alpha-1}(y^2-t^2)^{-1/2}dy\,,\qquad (|t|\le 1),$$

and

$$\lambda_\alpha=\frac{\Gamma(\alpha)}{2^{\alpha-2}\{\Gamma(\alpha/2)\}^2}\,.$$

We state the properties of U_α in the following lemma.

LEMMA 3.2. *Let $\alpha>0$. Then*
(i) *$U_\alpha(z)$ is even, continuous in $\mathbb{C}$, and $U_\alpha(x)=0$ for $|x|\le 1$.*
(ii) *As $\epsilon\to 0^+$,*

$$U'_\alpha(1+\epsilon)=-\alpha\sqrt{2\epsilon}+O(\epsilon).$$

(iii) *As $\epsilon\to 0^+$,*

$$U_\alpha(1+\epsilon)=-\alpha\,\frac{2\sqrt{2}}{3}\,\epsilon^{3/2}+O(\epsilon^2).$$

(iv) *$xU'_\alpha(x)$ is decreasing for $x>1$.*

PROOF OF LEMMA 3.2. See the proof of [10, Lemma 3.1, p. 71]. □

Now we are ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1. We distinguish two cases.

Case 1. $p=\infty$.
Let $f\in \mathrm{GANP}_n$. Since $\log f$ is subharmonic function, we may apply [12, Theorem 2.2, p. 208] to f as well. This idea is due to D. S. Lubinsky. From [12, Theorem 2.2, p. 208], with $a=a_n$ and a suitable substitution, we see that

$$(3.1)\qquad |f(x)W_\alpha(a_nx)|\le\| f(u)W_\alpha(a_nu)\|_{L^\infty([-1,1])}\exp(nU_\alpha(x)),$$

for $f\in\mathrm{GANP}_n$ and $|x|>1$.
Next, by [12, Theorem 2.6, p. 209], $U_\alpha(x)<0$ for $|x|>1$. Then, Theorem 2.1 follows from (3.1).

Case 2. $0<p<\infty$.
We closely follow the proof of [11, Lemma 7.4, p. 54]. By Lemma 3.2, we have

$$(3.2)\qquad U_\alpha(1+\eta)\sim-\eta^{3/2},\qquad 0<\eta<\eta_0\,,$$

and

$$U'_\alpha(1+\eta) \sim -\eta^{1/2}, \qquad 0 < \eta < \eta_0 \,.$$

Since $xU'_\alpha(x)$ is decreasing for $x > 1$,

$$U'_\alpha(x) \le -c_1\eta^{1/2}x^{-1}, \qquad x \ge 1+\eta, \qquad 0 < \eta < \eta_0 \,.$$

Integrating the above inequality, and using (3.2), we have, for $x \ge 1+\eta$, $0 < \eta < \eta_0$,

$$U_\alpha(x) \le -c_2\left\{\eta^{3/2} + \eta^{1/2}\log\left(\frac{x}{1+\eta}\right)\right\}.$$

Using this, we have, if δ_n is given as in Theorem 2.1,

$$\begin{aligned} I_{n,p} &= \left\{\int_{|x|\ge a_n(1+\delta_n)} \exp(npU_\alpha(x/a_n))dx\right\}^{1/p} \\ &\le c_3(n+1)^{-c_2K_n}a_n^{1/p} \,, \end{aligned}$$

for $n \ge \epsilon$, provided $K_n \ge C_1$ for some large enough $C_1 > 0$.
Next, if $f \in \mathrm{GANP}_n$ and $n \ge \epsilon$, then we obtain, from (3.1),

$$\begin{aligned} &\| fW_\alpha \|_{L^p(|x|\ge a_n(1+\delta_n))} \\ &\quad\le \| fW_\alpha \|_{L^\infty([-a_n,a_n])} I_{n,p} \\ &\quad\le (n+1)^{c_4-c_2K_n} \| fW_\alpha \|_{L^p(\mathbb{R})} \,, \end{aligned}$$

by Lemma 3.1. If

$$K_n \ge \frac{2c_4}{c_2}\,, \qquad n \ge \epsilon,$$

then we have

$$\| fW_\alpha \|_{L^p(|x|\ge a_n(1+\delta_n))} \le (n+1)^{-c_5K_n} \| fW_\alpha \|_{L^p(\mathbb{R})} \,,$$

hence, Theorem 2.1 follows. □

To prove Theorem 2.2, we need lemmas which are the consequences of [2, Theorem 4, p. 258]. First we restate [2, Theorem 4, p. 258].

Let

$$\mathrm{GANP}_n(s) \stackrel{\text{def}}{=} \{f \in \mathrm{GANP}_n : m(\{x \in [-1,1] : f(x) \le 1\}) \ge 2-s\},$$

where $0 < s < 2$. Then there exists an absolute constant C such that
(3.3)

$$f(x) \le \exp\left(Cn \cdot \min\left\{\frac{s}{\sqrt{1-x^2}}, \sqrt{s}\right\}\right), \quad 0 < s \le 1, \quad -1 < x < 1,$$

for all $f \in \mathrm{GANP}_n(s)$.

It is easy see that (3.3) implies the following lemma.

LEMMA 3.3. *Let $0 < \delta < 1$ and $d > 0$. Let*

$$s_n = \min\left\{\frac{d}{n}, 1\right\}, \quad n \in \mathbb{R}^+.$$

Then there exists a positive constant C^* depending on δ and d such that

$$\max_{|x|\le 1-\delta} f(x) \le C^*,$$

for all $f \in \mathrm{GANP}_n(s_n)$.

We can rewrite Lemma 3.3 as follows.

LEMMA 3.4. *Let $0 < \delta < 1$ and $d > 0$. Let*

$$s_n = \min\left\{\frac{d}{n}, 1\right\}, \quad n \in \mathbb{R}^+.$$

Then there exists a positive constant C depending on δ and d such that

$$m\left(\left\{y \in [-1,1] : Cf(y) \ge \max_{|x|\le 1-\delta} f(x)\right\}\right) \ge s_n\,,$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.

PROOF OF LEMMA 3.4. Let C^* be as in Lemma 3.3. We show that $C = 2C^*$ satisfies Lemma 3.4. Assume that Lemma 3.4 is not true with this $C = 2C^*$. Then there exists $f \in \mathrm{GANP}_n$ such that

$$m\left(\left\{y \in [-1,1] : 2C^* f(y) < \max_{|x|\le 1-\delta} f(x)\right\}\right) > 2 - s_n\,.$$

Define the function g by

$$g = 2C^* f / \max_{|x|\le 1-\delta} f(x).$$

Then $g \in \mathrm{GANP}_n(s_n)$ and

$$\max_{|x|\le 1-\delta} g(x) = 2C^* > C^*,$$

which is a contradiction to Lemma 3.3. □

As a consequence of Lemma 3.4, we have

LEMMA 3.5. *Let $0 < \delta < 1$ and $d > 0$. Let*

$$s_n = \min\left\{\frac{d}{n}, 1\right\}, \quad n \in \mathbb{R}^+.$$

Then there exists a positive constant C depending on δ and d such that for all measurable sets $D_n \subset [-1,1]$ with $m(D_n) \le s_n$,

$$\| f \|_{L^\infty([-1+\delta, 1-\delta])} \le C \| f \|_{L^\infty([-1,1]\setminus D_n)} ,$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.

Now we are ready to prove Theorem 2.2.

PROOF OF THEOREM 2.2. Let $\epsilon > 0$ and $d > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. Let

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}, \quad n \in \mathbb{R}^+.$$

We distinguish two cases.

Case 1. $p = \infty$. By Theorem 2.1, there exists $B^* > 1$ such that

$$\| fW_\alpha \|_{L^p(\mathbb{R})} \le 2 \| fW_\alpha \|_{L^p([-B^*a_n, B^*a_n])} , \qquad (0 < p < \infty), \tag{3.4}$$

for $f \in \mathrm{GANP}_n$, $n \ge \epsilon$. By [6, Theorem 1.1, p. 150], there exist polynomials $S_m \in \mathbb{P}_m$, $m \in \mathbb{N}$, and a constant $c^* > 0$ such that

$$S_m(x) \sim W_\alpha(x), \qquad \text{for } |x| \le c^* a_m .$$

Choose $k > 0$ so that $c^* k^{1/\alpha} \ge B^*$ and for each $n \in \mathbb{R}^+$, let $N = [kn]$. Define $S_0 \equiv 1$.
Then we have

$$S_N(x) \sim W_\alpha(x), \quad |x| \le B^* a_n .$$

Let $f \in \mathrm{GANP}_n$. Since $B^* > 1$, by Lemma 3.5, there exists a positive constant c_1, such that for all measurable sets $\Delta_n \subset [-B^*a_n, B^*a_n]$ with $m(\Delta_n) \le s_n$,

$$\| fS_N \|_{L^\infty([-a_n, a_n])} \le c_1 \| fS_N \|_{L^\infty([-B^*a_n, B^*a_n]\setminus \Delta_n)} .$$

Then by Theorem 2.1, we have

$$\begin{aligned}
\| fW_\alpha \|_{L^\infty(\mathbb{R})} &= \| fW_\alpha \|_{L^\infty([-B^*a_n,B^*a_n])} \\
&= \| fW_\alpha \|_{L^\infty([-a_n,a_n])} \\
&\leq c_2 \| fS_N \|_{L^\infty([-a_n,a_n])} \\
&\leq c_3 \| fS_N \|_{L^\infty([-B^*a_n,B*a_n]\setminus\Delta_n)} \\
&\leq c_4 \| fW_\alpha \|_{L^\infty([-B^*a_n,B^*a_n]\setminus\Delta_n)} . \qquad (3.5)
\end{aligned}$$

Case 2. $0 < p < \infty$.
Let $f \in \mathrm{GANP}_n$, $n \geq \epsilon$. Let

$$I_n(f) = \left\{ x \in [-B^*a_n, B^*a_n] : c_4^p f^p(x) W_\alpha^p(x) \geq \| fW_\alpha \|^p_{L^\infty([-B^*a_n,B^*a_n])} \right\} .$$

Then by (3.5), $m(I_n(f)) \geq s_n$.
Let $\Delta_n \subset [-B^*a_n, B^*a_n]$ with $m(\Delta_n) \leq s_n/2$ and let $A_n = I_n(f) \setminus \Delta_n$. Then $m(A_n) \geq s_n/2$, hence,

$$\begin{aligned}
\int_{\Delta_n} f^p(x) W_\alpha^p(x) dx &\leq \int_{\Delta_n} \| fW_\alpha \|^p_{L^\infty([-B^*a_n,B^*a_n])} dx \\
&\leq c_4^p \int_{A_n} f^p(x) W_\alpha^p(x) dx \\
&\leq c_4^p \int_{\substack{|x|\leq B^*a_n \\ x\notin\Delta_n}} f^p(x) W_\alpha^p(x) dx .
\end{aligned}$$

This together with Theorem 2.1 gives

$$\int_{-\infty}^{\infty} f^p(x) W_\alpha^p(x) dx \leq c_5 \int_{\substack{|x|\leq B^*a_n \\ x\notin\Delta_n}} f^p(x) W_\alpha^p(x) dx ,$$

for all $f \in \mathrm{GANP}_n$, $n \geq \epsilon$. □

Next we prove Theorem 2.3.

Proof of Theorem 2.3. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, and $0 < p < \infty$. By [6, Theorem 1.1, p. 150], there exist polynomials $S_m \in \mathbb{P}_m$, $m \in \mathbb{N}$, and a constant $c_1 > 0$ such that

$$S_m(x) \sim W_\alpha(x), \qquad \text{for } |x| \leq c_1 a_m .$$

Choose $k > 0$ so that $c_1 k^{1/\alpha} \geq 2$ and for each $n \in \mathbb{R}^+$, let $M = [kn]$. Define $S_0 \equiv 1$.

Then we have

$$S_M(x) \sim W_\alpha(x), \qquad |x| \leq 2a_n. \tag{3.6}$$

Then by (3.6), for $|x| \leq 2a_n$,

$$\begin{aligned}
\frac{\omega_{n,p}(W_\alpha;x)}{W_\alpha^p(x)} &\geq \inf_{f\in \mathrm{GANP}_n} \int_{-2a_n}^{2a_n} \frac{|f(t)W_\alpha(t)|^p}{|f(x)W_\alpha(x)|^p}dt \\
&\geq c_2 \inf_{f\in \mathrm{GANP}_n} \int_{-2a_n}^{2a_n} \frac{|f(t)S_M(t)|^p}{|f(x)S_M(x)|^p}dt \\
&\geq c_2 \inf_{f\in \mathrm{GANP}_{n+M}} \int_{-2a_n}^{2a_n} \frac{f^p(t)}{f^p(x)}dt \\
&\geq c_3 a_n \inf_{f\in \mathrm{GANP}_{n+M}} \int_{-1}^{1} \frac{f^p(t)}{f^p(\frac{x}{2a_n})}dt\ .
\end{aligned}$$

By [4, Theorem 3.1, p. 114], if $|x| \leq a_n$,

$$\inf_{f\in \mathrm{GANP}_{n+M}} \int_{-1}^{1} \frac{f^p(t)}{f^p(\frac{x}{2a_n})}dt \geq \frac{c_4}{n},$$

hence,

$$\omega_{n,p}(W_\alpha;x) \geq c_5 \frac{a_n}{n} W_\alpha^p(x), \quad |x| \leq a_n. \tag{3.7}$$

By Theorem 2.1, we have

$$\| fW_\alpha \|_{L^\infty(\mathbb{R})} \leq \| fW_\alpha \|_{L^\infty([-a_n,a_n])}, \quad f \in \mathrm{GANP}_n.$$

By the definition of $\omega_{n,p}(W_\alpha;x)$ and (3.7), we have, for all $f \in \mathrm{GANP}_n$,

$$\begin{aligned}
\| fW_\alpha \|_{L^\infty(\mathbb{R})} &\leq \| fW_\alpha \|_{L^\infty([-a_n,a_n])} \\
&\leq c_6 \left(\frac{n}{a_n}\right)^{\frac{1}{p}} \| fW_\alpha \|_{L^p(\mathbb{R})},
\end{aligned} \tag{3.8}$$

therefore, (3.7) holds for all $x \in \mathbb{R}$.

For the upper bound, we have

$$\begin{aligned}
\omega_{n,p}(W_\alpha;x) &= \inf_{f\in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{f^p(t)W_\alpha^p(t)}{f^p(x)}dt \\
&\leq \inf_{P\in \mathbb{P}_{[n]}} \int_{-\infty}^{\infty} \frac{|P(t)|^p W_\alpha^p(t)}{|P(x)|^p}dt \\
&= \lambda_{[n]+1,p}(W_\alpha;x),
\end{aligned}$$

hence, Theorem 2.3 is proved. □

PROOF OF THEOREM 2.4. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, and $0 < p < r \le \infty$. For $r = \infty$, we already proved Theorem 2.4 – see (3.8). Now suppose that $0 < p < r < \infty$. Then for $f \in \mathrm{GANP}_n$,

$$\begin{aligned} \| fW_\alpha \|_{L^r(\mathbf{R})}^r &= \int_{-\infty}^{\infty} |f(x)W_\alpha(x)|^{r-p}|f(x)W_\alpha(x)|^p dx \\ &\le \| fW_\alpha \|_{L^\infty(\mathbf{R})}^{r-p} \| fW_\alpha \|_{L^p(\mathbf{R})}^p \\ &\le c\left(\frac{n}{a_n}\right)^{\frac{r-p}{r}} \| fW_\alpha \|_{L^r(\mathbf{R})}^{r-p} \| fW_\alpha \|_{L^p(\mathbf{R})}^p , \end{aligned}$$

hence, Theorem 2.4 follows by taking pth root. □

PROOF OF THEOREM 2.5. Let $\epsilon > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, and $0 < p < r \le \infty$. Let

$$s = \frac{r}{p} > 1 \quad \text{and} \quad s' = \frac{s}{s-1} .$$

Let $f \in \mathrm{GANP}_n$, $n \ge \epsilon$. Then, by Theorem 2.1 and Hölder's inequality, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)W_\alpha(x)|^p dx &\le c_1 \int_{-c_2a_n}^{c_2a_n} |f(x)W_\alpha(x)|^p dx \\ &\le c_1 \left(\int_{-c_2a_n}^{c_2a_n} |f(x)W_\alpha(x)|^{ps} dx\right)^{\frac{1}{s}} \left(\int_{-c_2a_n}^{c_2a_n} dx\right)^{\frac{1}{s'}} \\ &= c_1 \left(\int_{-c_2a_n}^{c_2a_n} |f(x)W_\alpha(x)|^{r} dx\right)^{\frac{p}{r}} (2c_2a_n)^{\frac{r-p}{r}} \\ &\le c_3(a_n)^{\frac{r-p}{r}} \| fW_\alpha \|_{L^r(\mathbf{R})}^p . \end{aligned}$$

Taking pth root yields Theorem 2.5. □

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