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GENERALIZED SOBOLEV SPACE OF ROUMIEU TYPE AND SOME RELATED PROBLEMS

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ABSTRACT. We consider the relations between locally convex spaces of generalized Sobolev spaces of Roumieu type.

1. Introduction

The purpose of this paper is to investigate the relation between the generalized Sobolev space $W_{L^p}(\Omega;[M_k])$ of Roumieu type and the space $V_{L^p}(\Omega;[M_k])$ (see Definition. 3.2).

Also we show the space $D(\Omega;[M_k])$ coincides with the space $D_{L^p}(\Omega;[M_k])$ under some assumptions on a defining sequence of positive numbers (see Theorem 3.6).

Consequently we have the relations:

$$D(K;[M_k]) = D_{L^p}(K;[M_K]) = D(K;[M_{k+l}])$$

and

$$D(\Omega;[M_k]) = D_{L^p}(\Omega;[M_K]) = D(\Omega;[M_{k+l}])$$

if l is an integer greater than n/p and $\{M_k\}$ satisfies (M.1), (M.2)$'$ and (M.3)$'$ (see Theorem 3.8).

2. Locally convex space $D(\Omega;[M_k])$

Let $\{M_k\}$, $k \in N_0$, be a sequence of positive numbers which satisfies some of the following conditions:

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(M.1) (logarithmic convexity) $M_k^2 \leq M_{k-1}M_{k+1}, k \in N$;

(M.2) (stability under ultradifferential operators) There are constants $K > 0$ and $H > 1$ such that

$$M_k \leq KH^k \min_{0 \leq l \leq k} M_l M_{k-l}, k \in N_0;$$

(M.3) (strong non-quasi-analyticity) There is a constant $K > 0$ such that

$$\sum_{l=k+1}^{\infty} \frac{M_{l-1}}{M_l} \leq Kk \frac{M_k}{M_{k+1}}, \quad k \in N;$$

(M.2)′ (stability under differential operators) There are constants $K > 0$ and $H > 1$ such that

$$M_{k+1} \leq KH^k M_k, k \in N_0;$$

(M.3)′ (non-quasi-analyticity) $\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$

DEFINITION 2.1. Let K be a compact set in R^n, let $\{M_k\}$ be a sequence of positive numbers and let $h > 0$. We denote by $D(K; M_k, h)$ the space of all $f \in C^\infty(R^n)$ with support in K which satisfies (2.1).

$$\|f\|_{K,M_k,h} = \sup_{x \in K, \alpha \in N_0^n} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty \tag{2.1}$$

Clearly $D(K; M_k, h)$ is a Banach space under the norm (2.1).

PROPOSITION 2.2. *If $h < r$ and $K \Subset L$, then the inclusion mappings*

$$D(K; M_k, h) \longrightarrow D(K; M_k, r),$$
$$D(K; M_k, h) \longrightarrow D(L; M_k, h)$$

are compact operators.

PROOF. We can find a proof in Komatsu [1], p. 41. □

DEFINITION 2.3. Let K be a compact set in R^n and let Ω be an open set in R^n. As locally convex spaces we define:

$$D(K;[M_k]) = \text{ind} \lim_{n\to\infty} D(K;M_k,n),$$

$$D(\Omega;[M_k]) = \text{ind} \lim_{\substack{K\subset\subset\Omega \\ n\to\infty}} D(K;M_k,n)$$
$$= \text{ind} \lim_{K\subset\subset\Omega} D(K;[M_k]).$$

THEOREM 2.4. *$D(K;[M_k])$ and $D(\Omega;[M_k])$ are (DFS)-spaces.*

PROOF. By proposition 2.2 the theorem is clear. □

3. The generalized Sobolev spaces $W_{L^p}(\Omega;[M_k])$

Let Ω be an open set in R^n and let $1 \le p \le \infty$.

DEFINITION 3.1. Suppose that $\{M_k\}$ satisfies (M.2). Let $W_{L^p}(\Omega;(M_k))$ (resp. $W_{L^p}(\Omega;[M_k])$) be the space of all functions $u \in L^p(\Omega)$ such that for every $h > 0$ there exists $C = C(h) > 0$ (resp. there exist $h > 0$ and $C > 0$) satisfying

$$\|D^\alpha u\|_{L^p} \le Ch^{|\alpha|}M_{|\alpha|}, |\alpha| = 0, 1, 2\cdots. \tag{3.1}$$

We recall that Sobolev space $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m\}$ is a Banach space with the norm $\|u\|_{m,p} = \{\sum_{0\le|\alpha|\le m} \|D^\alpha u\|^p_{L^p(\Omega)}\}^{1/p}$ if $1 \le p < \infty$,

$$\|u\|_{m,\infty} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty.$$

Obviously $W^{o,p}(\Omega) = L^p(\Omega)$. In particular $W^{m,2}(\Omega)$ forms a Hilbert space with the inner product $(u,v) = \int_\Omega \sum_{0\le|\alpha|\le m} D^\alpha u(x)\overline{D^\alpha v(x)}dx$.

Clearly, we have

$$W_{L^p}(\Omega;(M_k)) \subset W_{L^p}(\Omega;[M_k]) \subset W^{m,p}(\Omega), \quad 1 \le p \le \infty. \tag{3.2}$$

We define, for $h > 0$,

(3.3)
$$W_{L^p}(\Omega; M_k, h) = \left\{ u \in L^p(\Omega); \|u\|_\infty^{p,h} = \sup_\alpha \left(\frac{\|D^\alpha u\|_{L^p}}{h^{|\alpha|} M_{|\alpha|}} \right) < \infty \right\}.$$

$W_{L^p}(\Omega; M_k, h)$ is a Banach space with the norm $\| \cdot \|_\infty^{p,h}$.
It is clear that

$$W_{L^p}(\Omega; [M_k]) = \cup_{h>0} W_{L^p}(\Omega; M_k, h)$$

(resp. $W_{L^p}(\Omega; (M_k)) = \cap_{h>0} W_{L^p}(\Omega; M_k, h)$.)
The imbedding, for $0 < h_1 < h_2$,

$$W_{L^p}(\Omega; M_k, h_1) \longrightarrow W_{L^p}(\Omega; M_k, h_2)$$

is continuous. Also $W_{L^p}(\Omega; M_k, h_1)$ is a closed subspace of $W_{L^p}(\Omega; M_k, h_2)$.

Hence we can define on $W_{L^p}(\Omega; [M_k])$ (resp. $W_{L^p}(\Omega; (M_k))$) the inductive (resp. projective) limit of the spaces $W_{L^p}(\Omega; M_k, h)$:

$$W_{L^p}(\Omega; [M_k]) = \operatorname{ind} \lim_{h \to \infty} W_{L^p}(\Omega; M_k, h) \tag{3.4}$$

$$(\text{resp.} \quad W_{L^p}(\Omega; (M_k)) = \operatorname{proj} \lim_{h \to 0} W_{L^p}(\Omega; M_k, h)) \tag{3.5}$$

The inductive (resp. projective) limit space $W_{L^p}(\Omega; [M_k])$ (resp. $W_{L^p}(\Omega; (M_k))$) is called to be the generalized Sobolev space of Roumieu (resp. Beurling) type.

DEFINITION 3.2. Suppose that $\{M_k\}$ satisfies (M.1) and (M.3)′. Let $1 < p < \infty$, $h > 0$ and let Ω be an open set in R^n.

We define

$$V_{L^p}(\Omega; M_k, h) = \{u \in L^p(\Omega); \|u\|_{p,h} < \infty\}, \tag{3.6}$$

$$\text{where} \quad \|u\|_{p,h} = \left(\sum_{|\alpha|=0}^{\infty} \left(\frac{\|D^\alpha u\|_{L^p}}{h^{|\alpha|} M_{|\alpha|}} \right)^p \right)^{1/p} \tag{3.7}$$

Then the space $V_{L^p}(\Omega; M_k, h)$ is a Banach space with the norm $\|\cdot\|_{p,h}$. The imbedding, for $0 < h_1 < h_2$,

$$V_{L^p}(\Omega; M_k, h_1) \subset V_{L^p}(\Omega; M_k, h_2)$$

is continuous. We define as locally convex spaces;

$$V_{L^p}(\Omega; [M_k]) = \text{ind} \lim_{h \to \infty} V_{L^p}(\Omega; M_k, h), \tag{3.8}$$

$$V_{L^p}(\Omega; (M_k)) = \text{prop} \lim_{h \to 0} V_{L^p}(\Omega; M_k, h). \tag{3.9}$$

THEOREM 3.3. *We have the following relations: For* $1 < p < \infty$, $W_{L^p}(\Omega; M_k, h) \subset V_{L^p}(\Omega; M_k, h+\eta) \subset W_{L^p}(\Omega; M_k, h+\eta)$ *for every* $\eta > o$, $W_{L^p}(\Omega; [M_k]) = V_{L^p}(\Omega; [M_k])$.

PROOF. We can show the relations easily. □

DEFINITION 3.4. Suppose that $\{M_k\}$ satisfies (M.1) and (M.3)$'$. Let $1 < p < \infty$, $h > 0$ and let K be a compact set in R^n. We define

$$D_{L^p}(K; [M_k], h) = \left\{ \phi \in D_K : \|\phi\|_{p,h}^{K,[M_k]} < \infty \right\} \tag{3.10}$$

where

$$\|\phi\|_{p,h}^{K,[M_k]} = \left(\sum_{|\alpha|=0}^{\infty} \left(\frac{\|D^\alpha \phi\|_{L^p(R^n)}}{h^{|\alpha|} M_{|\alpha|}} \right)^p \right)^{1/p}. \tag{3.11}$$

Then $D_{L^p}(K; [M_k], h)$ is a Banach space with the norm (3.11). If $h_1 < h_2$, then we have

$$D_{L^p}(K; [M_k], h_1) \subset D_{L^p}(K; [M_k], h_2)$$

and the inclusion mapping is continuous.

DEFINITION 3.5. Let Ω be an open set in R^n. Under the same assumptions as in Definition 3.4., we define as locally convex spaces

$$D_{L^p}(K;[M_k]) = \text{ind} \lim_{h\to\infty} D_{L^p}(K;[M_k],h), \tag{3.12}$$

$$D_{L^p}(\Omega;[M_k]) = \text{ind} \lim_{K\subset\subset\Omega} D_{L^p}(K;[M_k]). \tag{3.13}$$

We may also define

$$D_{L^p}(\Omega;[M_k],h) = \text{ind} \lim_{K\subset\subset\Omega} D_{L^p}(K;[M_k],h),$$

and

$$\begin{aligned} D_{L^p}(\Omega;[M_k]) &= \text{ind} \lim_{h\to\infty} D_{L^p}(\Omega;[M_k],h) \\ &= \text{ind} \lim_{\substack{K\subset\subset\Omega \\ h\to\infty}} D_{L^p}(K;[M_k],h). \end{aligned}$$

THEOREM 3.6. *Suppose that $\{M_k\}$ satisfies (M.1), (M.2)′ and (M.3)′. Then we have the following relations:*

$$\begin{aligned} D(K;M_k,h) &\subset D_{L^p}(K;[M_k],h+\eta) \quad \textit{for every} \quad \eta > 0, \\ D_{L^p}(K;[M_k],h) &\subset D(K;M_k,h+\eta) \\ &\qquad\qquad \textit{for every} \quad \eta \geq \rho h \quad \textit{for some} \quad \rho > 0, \\ D(K;[M_k]) &= D_{L^p}(K;[M_k]), \\ D(\Omega;[M_k]) &= D_{L^p}(\Omega;[M_k]). \end{aligned}$$

PROOF. If $\phi \in D(K;M_k,h)$, then $|D^\alpha\phi(x)| \leq Ah^{|\alpha|}M_{|\alpha|}, |\alpha| \in N_0$, for some constant $A > 0$. Therefore,

$$\|D^\alpha\phi\|_{L^p(R^n)} \leq Ah^{|\alpha|}M_{|\alpha|}|K|^{1/p},$$

where $|K|$ is the Lebesque measure of K.

For every $\eta > 0$ we can show that $\|\phi\|_{p,h+\eta}^{K,[M_k]} < \infty$.

If $\phi \in D_{L^p}(K;[M_k],h)$, then $\|D^\alpha\phi\|_{L^p(R^n)} \leq Bh^{|\alpha}M_{|\alpha|}, |\alpha| \in N_0$, for some positive constant B.

By Sobolev's theorem, there is a constant A_K depending on K such that, for $d > n/p(d \in N_0, 1 < p < \infty)$,

$$\|D^\alpha\phi\|_{C(K)} \leq A_K \sum_{|\beta|=d} \|D^{\alpha+\beta}\phi\|_{L^p}.$$

By (M.2)$'$, there are constants $C > 0$ and $H > 1$ such that

$$M_{|\alpha|+d} \leq C^d H^{d(|\alpha|+d)-\frac{d(d+1)}{2}} M_{|\alpha|}, \quad |\alpha| \in N_0.$$

Hence we have the following

$$\begin{aligned}
\|\phi\|_{K,M_k,h+\eta} &= \sup_{x\in K,\alpha} \frac{|D^\alpha\phi(x)|}{(h+\eta)^{|\alpha|}M_{|\alpha|}} \\
&\leq \sup_\alpha \frac{\|D^\alpha\phi\|_{C(K)}}{(h+\eta)^{|\alpha|}M_{|\alpha|}} \\
&\leq \sup_\alpha \frac{A_K \sum_{|\beta|=d} \|D^{\alpha+\beta}\phi\|_{L^p}}{(h+\eta)^{|\alpha|}M_{|\alpha|}} \\
&\leq A_K \binom{n+d-1}{d} \sup_\alpha \frac{\|D^{\alpha+\beta}\phi\|_{L^p}}{(h+\eta)^{|\alpha|}M_{|\alpha|}}
\end{aligned}$$

where $\max_{|\beta'|=d} \|D^{\alpha+\beta'}\phi\|_{L^p} = \|D^{\alpha+\beta}\phi\|_{L^p}$,

$$\begin{aligned}
&\leq BA_K h^d \binom{n+d-1}{d} \sup_\alpha \frac{h^{|\alpha|}M_{|\alpha|+d}}{(h+\eta)^{|\alpha|}M_{|\alpha|}} \\
&\leq BA_K h^d \binom{n+d-1}{d} C^d H^{d^2-\frac{d(d+1)}{2}} \sup_\alpha \Big(\frac{hH^d}{h+\eta}\Big)^{|\alpha|} < \infty,
\end{aligned}$$

for $\eta \geq h(H^d - 1)$. □

PROPOSITION 3.7. ([1], Prop. 8.4.). *If l is an integer greater than n/p, we have*

$$D(K;[M_k]) \subset D_{L^p}(K;[M_k]) \subset D(K;[M_{k+l}]),$$
$$D(\Omega;[M_k]) \subset D_{L^p}(\Omega;[M_k]) \subset D(\Omega;[M_{k+l}])$$

and the inclusion mappings are continuous.

THEOREM 3.8. *If l is an integer greater than n/p and $\{M_k\}$ satisfies (M.1), (M.2)′ and (M.3)′, then we have the followings:*

$$D(K;[M_k]) = D_{L^p}(K;[M_k]) = D(K;[M_{k+l}]),$$
$$D(\Omega;[M_k]) = D_{L^p}(\Omega;[M_k]) = D(\Omega;[M_{k+l}]).$$

PROOF. We can prove it by Theorem 3.6, Proposition 3.7 and the following Proposition 3.9. □

PROPOSITION 3.9. *Under the same assumption as in* Theorem 3.8, *we have the following relation:*

$$D(K;M_{k+l},h) \subset D_{L^p}(K;[M_k],h+\eta)$$

for every $\eta \geq \rho h$ for some $\rho > 0$.

PROOF. Let $\phi \in D(K;M_{k+l},h)$. By (M.2)′, there exist constants $C > 0$ and $H > 1$ such that $M_{|\alpha|+l} \leq C^l H^{l(|\alpha|+l)-\frac{l(l+1)}{2}} M_{|\alpha|}$.

There exists constants $A > 0$ such that $\|D^\alpha \phi\|_{L^p(R^n)} \leq A h^{|\alpha|} M_{|\alpha|+l}$ $|K|^{1/p}$, $|\alpha| \in N_0$, where $|K|$ is the Lebesgue measure of K.

Therefore,

$$\begin{aligned}
\|\phi\|_{p,h+\eta}^{K,[M_k]} &= \left(\sum_{|\alpha|=0}^{\infty} \left(\frac{\|D^\alpha \phi\|_{L^p(R^n)}}{(h+\eta)^{|\alpha|} M_{|\alpha|}} \right)^p \right)^{1/p} \\
&\leq A|K|^{1/p} \left(\sum_{|\alpha|=0}^{\infty} \left(\frac{h^\alpha M_{|\alpha|+l}}{(h+\eta)^{|\alpha|} M_{|\alpha|}} \right)^p \right)^{1/p} \\
&\leq A|K|^{1/p} C^l H^{l^2 - \frac{l(l+1)}{2}} \left(\sum_{|\alpha|=0}^{\infty} \left(\frac{hH^l}{h+\eta} \right)^{|\alpha|p} \right)^{1/p} < \infty,
\end{aligned}$$

if $\eta > h(H^l - 1)$.

□

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