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ON GALOIS GROUPS FOR NON-IRREDUCIBLE INCLUSIONS OF SUBFACTORS

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ABSTRACT. We apply sector theory to obtain some characterization on Galois groups for subfactors. As an example of a non-irreducible inclusion of small index, a locally trivial inclusion arising from an automorphism is considered and its Galois group is completely determined by using sector theory.

1. Introduction

Index theory for II_1-factors was started by Jones in [8] and it was extended to properly infinite factors by Kosaki in [10]. Index theory has played a fundamental role to classify subfactors and automorphisms of subfactors. Especially, Galois groups have been studied by many authors (see for example [2,3,11,13]).

The concept of sector (or equivalently bimodule) was initially considered by Ocneanu's work [16] and Longo's approach on index theory for properly infinite factors [15]. The sector technique is very useful to attack problems related to the Galois group. For an irreducible inclusion case, the precise description of Galois group was obtained (see [11,12]). In fact, for an irreducible and finite index inclusion $N \subset M$ case, the Galois group is just the group of automorphisms appearing in the decomposition of the canonical endomorphism $\rho\bar{\rho}$ associated to a given inclusion $N = \rho(M) \subset M$.

It is natural to consider Galois groups for non-irreducible inclusions. But for a non-irreducible inclusion case, automorphisms given as above do not necessarily form a group, so characterization given as above is actually false and it is difficult to describe its Galois group completely.

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The purpose of the present paper is to characterize Galois groups for subfactors in the sense of sectors and give some properties on them. As examples of non-irreducible inclusions, we consider Jones tower of II_1-factors and a locally trivial inclusion of factors arising from an automorphism. In §2 we will introduce Jones' index, sector theory, and Galois group for subfactors. In §3 we will use sector theoretic approach and give some results on Galois groups. In §4 we will consider a locally trivial inclusion of index 4 and describe its Galois group by applying sector theory.

2. Preliminaries

In this section we summarize definitions and basic facts on sectors and Galois groups for our later purpose. Further details can be found in [1,5,9].

For a pair $N \subset M$ of II_1-factors, Jones' index $[M : N]$ is defined in [8] by the coupling constant $dim_N L^2(M)$ of N on the standard Hilbert space $L^2(M)$ with modular conjugation J_M. The von Neumann algebra $M_1 = J_M N' J_M$ is called the basic construction of $N \subset M$ and by iterating the basic constructions we get Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$. For a properly infinite factors we use Kosaki (minimal) index in [10]. If the relative commutant $N' \cap M$ is trivial then the inclusion $N \subset M$ is called an irreducible inclusion.

For a properly infinite factor M, $End(M)$ denotes the set of unital, normal endomorphisms of M. For a given $\rho \in End(M)$, let $H_\rho = {}_M L^2(M)_M$ be the Hilbert space with M-M bimodule action $m_1 \cdot \xi \cdot m_2 = \rho(m_1) J_M m_2^* J_M \xi$. Two M-M bimodules H_{ρ_1} and H_{ρ_2} are unitarily equivalent if and only if ρ_1 and ρ_2 are inner conjugate. Any abstract M-M bimodule is unitary equivalent to H_ρ for some $\rho \in End(M)$.

We set $Sect(M) = End(M)/Int(M)$, and its class is called a sector, denoted by $[\rho]$ (or simply by ρ), where $Int(M)$ is the set of inner automorphisms. Note that $Sect(M)$ is the same as the set of M-M bimodules up to unitarily equivalent and ordinary composition of sectors corresponds to the relative tensor product of bimodules. For $\rho_1, \rho_2 \in End(M)$, choose isometries $v_1, v_2 \in M$ satisfying that $v_1 v_1^*$ and $v_2 v_2^*$ are orthogonal projections with $v_1 v_1^* + v_2 v_2^* = 1$. If we define ρ by

$\rho(x) = v_1\rho_1(x)v_1^* + v_2\rho_2(x)v_2^*$, $x \in M$, then we have $[\rho] = [\rho_1] \oplus [\rho_2]$ and denote it by $[\rho_i] \prec [\rho]$, $i = 1, 2$.

Moreover we know $End(H_\rho) = M \cap \rho(M)'$ and so H_ρ is irreducible if and only if $\rho(M) \subset M$ is irreducible. When $\rho(M) \subset M$ is of finite index, the relative commutant $\rho(M)' \cap M$ is finite dimensional so that by choosing minimal orthogonal projections $\{p_i\}_{i=1,\cdots,m}$ with $\sum p_i = 1$, we can obtain the irreducible decomposition $[\rho] = [\rho_1] \oplus \cdots \oplus [\rho_m]$ of ρ.

The square root $d\rho$ of index of $\rho(M) \subset M$ is called the statistical dimension of ρ and it satisfies that $d(\rho_1 \oplus \rho_2) = d\rho_1 + d\rho_2$ and $d(\rho_1\rho_2) = d\rho_1 \cdot d\rho_2$. Note that $d\rho = 1$ if and only if ρ is an automorphism of M (i.e., $\rho \in Out(M) = Aut(M)/Int(M)$).

For a given pair $N \subset M$ of factors, by tensoring an infinite factor $B(H)$ we get properly infinite isomorphic factors $N \otimes B(H) \subset M \otimes B(H)$. It is well known that for a given $\theta \in Aut(M, N)$ we get $\tilde{\theta} = \theta \otimes id \in Aut(M \otimes B(H), N \otimes B(H))$ and the relative commutant is invariant by this tensoring. So we may assume that $N \subset M$ is an inclusion of isomorphic properly infinite factors with finite index and ρ is an endomorphism of M with $\rho(M) = N$. However it is actually valid for any pair of factors.

The conjugate sector $\overline{[\rho]}$ of $[\rho]$ is given by $[\bar{\rho}]$ with $\bar{\rho} = \rho^{-1} \circ \gamma$, where γ is the canonical endomorphism for the inclusion $\rho(M) \subset M$ (see [14,15]).

Note that $H_{\rho\bar{\rho}}$ is unitarily equivalent to ${}_ML^2(M_1)_M$ via the unitary J_MJ_N and the decomposition rule for $1 \to \rho \to \rho\bar{\rho} \to \rho\bar{\rho}\rho \to \cdots$ corresponds to the dual principal graph for $N \subset M$.

Now we recall Galois group theory for subfactors (see [9]). The Galois group $G(M, N)$ for an inclusion $N \subset M$ of factors is defined by $G(M, N) = \{\alpha \in Aut(M) | \ \alpha|_N = id_N\}$ and the set of unitaries in M is denoted by $\mathcal{U}(M)$. The well known facts on Galois groups for subfactors are as follows:

(1) For any $\sigma \in Aut(M)$ we have $G(M, \sigma(N)) = \sigma G(M, N)\sigma^{-1}$.
(2) $G(M, N) \cap Int(M) \cong \mathcal{U}(N' \cap M)/\mathbb{T}$, where $\mathbb{T}$ denotes a one-dimensional torus. So if $N' \cap M = \mathbb{C} \cdot 1$ then G acts outerly on M.

When a finite group G acts outerly on a II_1-factor M, for a subgroup

H of G, we have the followings:

(1) $G(M^H, M^G) \cong G_0$, where $G_0 = \{g \in G |\ gHg^{-1} = H\}$. So we have $G(M, M^G) \cong G$.

(2) The Galois group $G(M \rtimes G, M \rtimes H)$ is isomorphic with the group of one-dimensional representation χ of G with $\chi|_H = 1$. So we have $G(M \rtimes G, M) \cong G/[G, G]$.

3. Some properties on Galois groups analyzed from sectors

To study sector theory is equivalent to that of bimodules for von Neumann algebra which was introduced by Connes (see [6,12] for the details). Thanks to Frobenius reciprocity, there is always a canonical pairing between $End(H \otimes H^*)$ and $End(H^* \otimes H)$, where $H = {}_N L^2(M)_M$, N-M bimodule. In the case of depth 2, this pairing provides Hopf structure to $End(H \otimes H^*)$ and its dual $End(H^* \otimes H)$ as well (see [4,19] for details). In terms of bimodules, the Galois group for $N \subset M$ can be formulated as the following.

Proposition 3.1. ([12]) *For an irreducible inclusion $N \subset M$ of factors, the irreducible bimodules of dimension 1 (i.e., automorphisms) appearing in the irreducible decomposition of the M-M bimodule ${}_M L^2(M_1)_M (= {}_M L^2(M) \otimes_N L^2(M)_M) = End(H^* \otimes H))$ form a finite group and this group is exactly the Galois group $G(M, N)$.*

For a properly infinite case, the above result means that $G(M, N)$ consists of automorphisms appearing in Longo's canonical endomorphism γ attached to $N \subset M$ (see [14]).

Now we describe Galois group $G(M, N)$ for the finite index inclusion $\rho(M) = N \subset M$, $\rho \in End(M)$, of factors by using sector theory. Note that for an irreducible sector $[\rho]$, we have $[id] \prec [\rho\bar{\rho}]$ and $[id] \prec [\bar{\rho}\rho]$. It is not necessary to assume irreducibility of ρ in the next results.

Lemma 3.2. *If $\alpha \in G(M, N)$ then $[\alpha\rho] = [\rho]$ and $[\alpha] \prec [\rho\bar{\rho}]$ hold.*

Proof. The fact that $\rho(x) \in N$, $\forall x \in M$, implies that for a given $\alpha \in G(M, N)$, $\alpha\rho(x) = \rho(x)$. So we have $[\alpha\rho] = [\rho]$. By using irreducible decomposition of ρ and Frobenius reciprocity, we have $[id] \prec [\rho\bar{\rho}]$ and $[\alpha] = [\alpha \circ id] \prec [\alpha\rho\bar{\rho}] = [\rho\bar{\rho}]$. $\square$

Here we set G_0 and $\tilde{G}$ as follows:

$$G_0 = \{[\alpha] \mid d\alpha = 1, \alpha \in Aut(M), [\alpha] \prec [\rho\bar{\rho}]\},$$

$$\tilde{G} = G(M,N)/G(M,N) \cap Int(M).$$

In the case of an irreducible pair $N \subset M$ of factors, it is known that G_0 is a group and Galois group $G(M,N)$ is isomorphic to G_0 (see [11,12]). But without assuming of irreducibility, we obtain the following theorem.

THEOREM 3.3. *For an inclusion $N \subset M$ of factors with $dim(N' \cap M) = k$, we have*

$$|\tilde{G}| \leq [M : N] - (k-1).$$

PROOF. When $\alpha \in G(M,N)$ we have $\alpha\rho = \rho$ and $d\rho = 1$. Since $[id] \prec [\rho\bar{\rho}]$, we get $[\alpha] = [\alpha \circ id] \prec [\alpha\rho\bar{\rho}] = [\rho\bar{\rho}]$ and so $[\alpha] \in G_0$. If we consider $\phi : G(M,N) \to G_0$ defined by $\phi(\alpha) = [\alpha]$ then ϕ induces an injection $\tilde{\phi} : \tilde{G} \to G_0$. Hence we have $|\tilde{G}| \leq |G_0|$ and $|G_0| \leq d(\rho\bar{\rho}) = (d\rho)^2 = [M : N]$.

From the irreducible decomposition of ρ, we know that id occurs in $\rho\bar{\rho}$ with multiplicity m with $m \geq k$ and so it follows $|G_0| \leq d(\rho\bar{\rho}) - k + 1$. Hence we get $|\tilde{G}| \leq [M : N] - (k-1)$. □

As an application of the preceding theorem, we obtain that Galois group $G(M,N)$ for an inclusion of factors is always a compact Lie group. Now consider that what happens for an irreducible inclusion. Since $G(M,N) \cap Int(M) \cong \mathcal{U}(N' \cap M)/\mathbb{T}$, we have $\tilde{G} = G(M,N)$. The preceding theorem says that $|G(M,N)| \leq [M : N]$ for an irreducible inclusion $N \subset M$.

For an inclusion $N \subset M$ of II_1-factors with $[M : N] < 4$, it is well known that the principal graph and the dual principal graph are same one of Coxeter graphs $A_n (n \geq 3), D_{2n} (n \geq 2), E_6$, and E_8 (see [5,6]). It is a folk result among specialists that the Galois group for this inclusion is closely related to Coxeter graphs. For example, if it is $D_{2n} (n \geq 3)$ and E_8 then $G(M,N) = \{id\}$. If it is A_3 (resp. D_4) then $G(M,N) \cong \mathbb{Z}_2$ (resp. $\mathbb{Z}_3$).

But for a non-irreducible inclusion $N \subset M$ of factors, we know that in many cases Galois group $G(M,N)$ consists of inner automorphisms.

PROPOSITION 3.4. *When $G(M,N)$ consists of inner automorphisms, if $\alpha \in Aut(M)$ with $\alpha(N) = N$ is outer then $\alpha|_N$ is an outer automorphism of N.*

PROOF. Suppose that $\alpha|_N \in Aut(N)$ is inner and $v \in \mathcal{U}(N)$ with $\alpha|_N = Adv$. Then we have $Adv^* \circ \alpha \in G(M,N)$ and from the assumption, $Adv^* \circ \alpha$ is an inner automorphism. Since $v^* \in \mathcal{U}(M)$, α is also an inner automorphism of M, which is a contradiction. □

The following lemma gives sufficient conditions guaranteeing that $G(M,N) \subset Int(M)$.

LEMMA 3.5. *For a non-irreducible inclusion $N = \rho(M) \subset M$ and irreducible decomposition $\rho = \rho_1 \oplus \rho_2 \cdots \oplus \rho_n$, if $d\rho_1 \neq d\rho_i$, $i = 2, \cdots, n$, and $G(M, \rho_1(M)) = \{id\}$ then $G(M,N)$ consists of inner automorphisms.*

PROOF. For any $\alpha \in G(M,N)$, by Lemma 3.2, we have $[\alpha\rho] = [\rho]$ and

$$[\alpha\rho_1] \oplus \cdots \oplus [\alpha\rho_n] = [\rho_1] \oplus \cdots \oplus [\rho_n].$$

Since $(\alpha\rho_i(M))' \cap M = \alpha(\rho_i(M)' \cap M) = \alpha(\mathbb{C} \cdot 1) = \mathbb{C} \cdot 1$, $\alpha\rho_i$ is irreducible and $d(\alpha\rho_i) = d\rho_i$. By the assumption, we get $[\alpha\rho_1] = [\rho_1]$. Since $[id] \prec [\rho_1\overline{\rho_1}]$, we have $[\alpha] = [\alpha \circ id] \prec [\alpha\rho_1\overline{\rho_1}] = [\rho_1\overline{\rho_1}]$. So $[\alpha]$ is an irreducible sector with $d\alpha = 1$ appearing in the irreducible decomposition of $\rho_1\overline{\rho_1}$. This implies that α can be adjusted in $G(M, \rho_1(M)) = \{id\}$ and so α is an inner automorphism. Therefore $G(M,N)$ consists of inner automorphisms. □

As a typical example of a non-irreducible inclusion, we will consider an irreducible pair of hyperfinite II_1-factors and downward Jones tower.

EXAMPLE 3.6. Now we consider $N = B_k \subset A = M$ for an irreducible pair $B \subset A$ of hyperfinite II_1-factors with $[A : B] < 4$ and downward Jones tower $A \supset B \supset B_1 \supset B_2 \supset \cdots$. We have already noted its Coxeter graphs. If this irreducible inclusion $B \subset A$ has Galois group $G(A,B) = \{id\}$, then from the Bratteli diagram and Lemma 3.5, we know that $G(A, B_1)$ consists of inner automorphisms. If $B \subset A$ has Coxeter graphs $D_{2n} (n \geq 3)$, E_8, $A_n, (n \geq 4)$, and E_6 then it is well known that $G(A,B_k) = \mathcal{U}(A \cap B_k')/\mathbb{T}$. Hence $G(A,B_k)$ consists of just inner automorphisms of A_k.

But if $B \subset A$ has Coxeter graphs A_3 and D_4 then $G(A,B_k)$ contains outer automorphisms (see [6,12]).

In fact, if $B \subset A$ has Coxeter graph A_3 (resp. D_4) then $B = A^{\mathbb{Z}_2}$ (resp. $B = A^{\mathbb{Z}_3}$). In general, we will get the following theorem for an outer action on a factor.

THEOREM 3.7. *For an outer action α of a finite group G on a II_1-factor A, if we consider a pair $B = A^{\alpha(G)} \subset A$ of II_1-factors and downward Jones tower $A \supset B \supset B_1 \supset B_2 \supset \cdots$, then $G(A,B_k), k = 1,2,\cdots$, contains outer automorphisms.*

PROOF. For an endomorphism ρ (resp. ρ_k) with $\rho(A) = B$ (resp. $\rho_k(A) = B_k$), we have $\rho\bar{\rho} = \sum_{g\in G} \alpha_g$ and $\rho_k = (\rho\bar{\rho})^m$ or $\rho_k = (\rho\bar{\rho})^m\rho$. So we have $\rho_k\overline{\rho_k} = (\rho\bar{\rho})^{2m} = \sum_{g\in G}^{\oplus} 2m\alpha_g$ or $\rho_k\overline{\rho_k} = (\rho\bar{\rho})^{2m+1} = \sum_{g\in G}^{\oplus}(2m+1)\alpha_g$. In any case, we have $\alpha_h\rho_k = \rho_k$, $\forall h \in G$ and $\alpha_h \in G(A,B_k)$, outer automorphisms of A, as desired. □

We now close this section with some properties on Galois groups for Jones tower. For Jones tower $N \subset M = M_0 \subset M_1 \subset \cdots$ of II_1-factors and $i \geq j \geq 0$, we have $G(M_i, M_{i-j}) \cong G(M_{i+2j}, M_{i+j})$. If we consider the basic construction $M^G \subset M \subset M \rtimes G$, where a finite non-abelian group G acts outerly on II_1-factor M. Then we know that $G(M, M^G)$ and $G(M \rtimes G, M)$ are not isomorphic. So $G(M_i, M_{i-1})$ and $G(M_{i+1}, M_i)$ may not be isomorphic.

Conversely, for an irreducible inclusion $N \subset M$ of II_1-factors, the Galois group $G(M,N)(= G)$ acts outerly on M and we get II_1-factors M^G and $M \rtimes G$ with $N \subset M^G \subset M \subset M \rtimes G$.

4. The Galois group for a locally trivial inclusion

As an example of a non-irreducible inclusion with the smallest index, we will consider "a locally trivial inclusion" arising from an automorphism.

Two pairs of factors $N \subset M$ and $B \subset A$ are called conjugate if there exists an isomorphism ϕ from M to A satisfying $\phi|_N$, an isomorphism from N to B. In this case, we get $G(M,N) = \phi^{-1}G(A,B)\phi$, which means that Galois groups are conjugacy invariant.

For an irreducible pair of II_1-factors and an automorphism θ, we first give a sufficient condition in order that θ can be adjusted to an element in the Galois group. The proof of the next proposition is a straightforward computation and left to the reader.

PROPOSITION 4.1. *Let $N \subset M$ be an irreducible pair of II_1-factors and θ an automorphism of M. If there exists a non-zero $x \in M$ such that $x\theta(y) = yx$, $\forall y \in N$, then θ can be adjusted to an element in $G(M,N)$.*

Note that for an inclusion of II_1-factors with Jones' index $[M : N] < 4$, it is an irreducible and its Galois group is isomorphic to $\{id\}, \mathbb{Z}_2$ or $\mathbb{Z}_3$. So it is natural to consider an inclusion $N \subset M$ of II_1-factors with $[M : N] = 4$ and $N' \cap M = \mathbb{C} \oplus \mathbb{C}$.

From now on, we shall determine the Galois group for the following inclusion $N \subset M$ arising from an automorphism $\alpha \in Aut(P)$, where P is hyperfinite II_1-factor.

$$N = \left\{ \begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix} \mid x \in P \right\} \subset P \otimes M_2(\mathbb{C}) = M.$$

For this inclusion, Jones' index $[M : N] = 4$ and $N' \cap M \cong \mathbb{C} \oplus \mathbb{C}$ are well known (see [12,20]). Moreover the complete list of subfactors with index less than or equal to 4 of hyperfinite II_1-factors was obtained (see [18]). By standard tensoring technique, the sector theory is valid for this pair $N \subset M$.

If we choose isometries $v_1, v_2 \in P$ with $v_1v_1^* + v_2v_2^* = 1$ and define

$$\rho(x) = v_1xv_1^* + v_2\alpha(x)v_2^*, \ x \in P,$$

then we have $\rho \in End(P)$ with $[\rho] = [id] \oplus [\alpha]$.

Here two inclusions $\rho(P) \subset P$ and $N \subset M$ are conjugate via ϕ defined by

$$\phi(x) = \begin{pmatrix} v_1^*xv_1 & v_1^*xv_2 \\ v_2^*xv_1 & v_2^*xv_2 \end{pmatrix}, \ x \in P.$$

This isomorphism allows us to compute $G(M,N)$.
If $\theta \in G(P,\rho(P))$ then we have $[\theta\rho] = [\rho]$ and

$$[\theta] + [\theta\alpha] = [id] + [\alpha].$$

Since θ and α are automorphisms of a factor P, $[\theta\alpha]$ is an irreducible sector. Hence two cases occur. One is $[\theta] = [\alpha]$ and $[\theta\alpha] = [id]$ with $[\alpha^2] = [id]$. The other is $[\theta] = [id]$ and $[\theta\alpha] = [\alpha]$. For this pair $N \subset M$ of hyperfinite II_1-factors, we are ready to prove the following two lemmas.

LEMMA 4.2.

(1) *α cannot be adjusted to an element in $G(M,N)$.*
(2) *If α is inner then $G(M,N)$ is isomorphic to $\mathbb{T}$.*

PROOF.

(1) Suppose that α can be adjusted to an element in $G(M,N)$. Then $[\alpha\rho] = [\rho]$. Since $[\rho\bar{\rho}] = 2[id] + [\alpha] + [\bar{\alpha}]$, we have $2[\alpha] \prec [\alpha\rho\bar{\rho}] = [\rho\bar{\rho}]$ and $2[\bar{\alpha}] \prec [\rho\bar{\rho}]$. So we get $4 = d(\rho\bar{\rho}) \geq 2d\alpha + 2d\bar{\alpha} + 2d(id) = 6$, which is a contradiction.
(2) If α is inner then $[\rho] = 2[id]$. So for a given $\theta \in G(M,N)$, $[\theta\rho] = [\rho]$ implies $2[\theta] = 2[id]$ and θ is inner. Since $N' \cap M = \mathbb{C} \oplus \mathbb{C}$, we have

$$G(M,N) = G(M,N) \cap Int(M) = \mathcal{U}(N' \cap M)/\mathbb{T} \cong \mathbb{T}. \qquad \square$$

Now we recall Connes outer conjugacy invariants p and γ in [3]. Two automorphisms α and β of a von Neumann algebra M are called outer conjugate when $[\alpha]$ and $[\beta]$ are conjugate in $Out(M) = Aut(M)/Int(M)$. For an automorphism α of M, we define outer period $p(\alpha) \in \mathbb{N}$ of α and Connes obstruction $\gamma(\alpha) \in \mathbb{C}$ of α by $\{n \in \mathbb{Z} | \ \alpha^n \in Int(M)\} = p(\alpha)\mathbb{Z}$ and $\alpha(u) = \gamma(\alpha)u$, for some $u \in \mathcal{U}(M)$ with $\alpha^{p(\alpha)} = Adu$. Teruya and Watatani determined the structure of lattices of intermediate subfactors for this inclusion in [20]. But by applying sector theory, we can give another proof of their theorem. More precisely, when $p(\alpha) = 2$ and $\gamma(\alpha) = -1$, there is no non-trivial intermediate factor of $N \subset M$.

LEMMA 4.3.

(1) *If $p(\alpha) = 2$ then we have $G(M,N) \cong \mathbb{Z}_2 \times \mathbb{T}$.*
(2) *If $p(\alpha) \neq 2$ then we have $G(M,N) \cong \mathbb{T}$.*

PROOF.

(1) If $p(\alpha) = 2$ then for a given $\theta \in G(P, \rho(P))$, we have $[\theta] = [\alpha]$ and $[\alpha^2] = [id]$.

Since P is a hyperfinite factor, we can choose v_1, v_2 in the fixed point algebra $P^\alpha(\cong P)$. By an inner perturbation of a unitary $w = e^{i\phi}v_1v_1^* + e^{-i\phi}v_2v_2^*$, $\phi \in \mathbb{R}$, we may assume $\alpha^2 = id$. Since $\theta = Adu \circ \alpha$, for some $u \in \mathcal{U}(P)$, we get $Adu \circ \alpha\rho = \rho$ and

$$u(v_1\alpha(x)v_1^* + v_2xv_2^*)u^* = v_1xv_1^* + v_2\alpha(x)v_2^*.$$

By a direct computation, this follows $v_1^*uv_1 = 0$, $v_1^*uv_2 = e^{i\psi}$, $v_2^*uv_1 = (v_1^*uv_2)^* = e^{-i\psi}$, $v_2^*uv_2 = 0$ and

$$u = e^{i\psi}v_1v_2^* + e^{-i\psi}v_2v_1^*, \ \psi \in \mathbb{R}.$$

So when outer period of α is 2, Galois group $G(P, \rho(P))$ is generated by

$$\{Ad(e^{i\phi}v_1v_1^* + e^{-i\phi}v_2v_2^*)|\phi \in \mathbb{R}\}$$
$$\cup \{Ad(e^{i\psi}v_1v_2^* + e^{-i\psi}v_2v_1^*) \circ \alpha|\psi \in \mathbb{R}\}.$$

Hence equivalently, Galois group $G(M, N)$ is generated by

$$\left\{Ad\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix}|\phi \in \mathbb{R}\right\}$$
$$\cup \left\{\alpha \otimes Ad\begin{pmatrix} 0 & e^{i\psi} \\ e^{-i\psi} & 0\end{pmatrix}|\psi \in \mathbb{R}\right\},$$

which is isomorphic to $\mathbb{Z}_2 \times \mathbb{T}$.

(2) If $p(\alpha) \neq 2$ then for a given $\theta \in G(P, \rho(P))$, $[\theta] = [id]$ and $[\theta\alpha] = [\alpha]$ hold.

So if we let $\theta = Adu$, $u \in \mathcal{U}(P)$, then we get $Adu \circ \rho = \rho$ and $u(v_1xv_1^* + v_2\alpha(x)v_2^*)u^* = v_1xv_1^* + v_2\alpha(x)v_2^*$. It is straightforward to see that

$$u = e^{i\psi}v_1v_1^* + e^{-i\psi}v_2v_2^*, \ \psi \in \mathbb{R},$$

and $G(M, N) \cong G(P, \rho(P)) \cong \mathbb{T}$. □

By summing up the computation so far, we now conclude the following result.

THEOREM 4.4. *In the above situation, α cannot be adjusted to an element in $G(M, N)$. When α^2 is inner, we have $G(M,N)/G(M,N) \cap Int(M) \cong \mathbb{Z}_2$ and $|\tilde{G}| = 2$. When α^2 is not inner (or α is inner), $G(M, N)$ consists of inner automorphisms and $|\tilde{G}| = 1$.*

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