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ON THE SUPERSTABILITY OF THE FUNCTIONAL EQUATION $\mathbf{f(x_1 + \cdots + x_m) = f(x_1) \cdots f(x_m)}$

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Abstract. First, we shall improve the superstability result of the exponential equation $f(x+y) = f(x)f(y)$ which was obtained in [4]. Furthermore, the superstability problems of the functional equation $f(x_1 + \cdots + x_m) = f(x_1) \cdots f(x_m)$ shall be investigated in the special settings (2) and (9).

1. Introduction

In 1940, S. M. Ulam [7] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

> *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

In the next year, D. H. Hyers [3] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. The result of Hyers was further generalized by Th. M. Rassias [6]. Since then, the stability problems of several functional equations have been extensively investigated. The terminology *Hyers–Ulam–Rassias stability* originates from this historical background (cf. [5]).

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In 1979, J. Baker, J. Lawrence and F. Zorzitto [2] proved a new type of stability of the exponential equation $f(x+y) = f(x)f(y)$ (see also [1]). More precisely, they proved that if a complex-valued mapping f defined on a normed vector space satisfies the inequality $|f(x+y) - f(x)f(y)| \leq \delta$ for some given $\delta > 0$ and for all x, y, then either f is bounded or f is exponential. Such a phenomenon is called the *superstability* of the exponential equation.

In this work, we consider the superstability of the functional equation $f(x_1 + \cdots + x_m) = f(x_1) \cdots f(x_m)$ which is explained as follows. For given $0 < p \leq q$, and $\varepsilon > 0$, we let f be a complex-valued mapping defined on a normed vector space V such that

$$|f(x+y) - f(x)f(y)| \leq \varepsilon \|x\|^p \|y\|^q \tag{1}$$

for all x, y in V. Now, let us define

$$\psi_{p,q}(x) = \max\{t > 0 : t - t^{1/2} - \varepsilon\|x\|^{p+q} = 0\}$$

for all $x \in V$. We remark that $\psi_{p,q}(x) > 1$ holds for every $x \neq 0$ (and $m \geq 2$).

We introduce the main results of this work, Theorem 2 and Theorem 3 whose proofs are presented in section 2 and section 3 respectively.

THEOREM 2. *It holds either* $|f(x)| \leq \max\{2^p, \psi_{p,q}(x)\}$ *for any* x *in* V, *where* $\psi_{p,q}(x) = \left(1 + 2\varepsilon\|x\|^{p+q} + \sqrt{1 + 4\varepsilon\|x\|^{p+q}}\right)/2$, *or else* $f(x+y) = f(x)f(y)$ *for every* $x, y \in V$.

In section 3, let m (> 2) be an integer and let $p_1, ..., p_m$ be given such that $0 < p_1 \leq \cdots \leq p_m$. Suppose that V is a normed vector space and that f is a complex-valued mapping defined on V and satisfying the following functional inequality

$$|f(x_1 + \cdots + x_m) - f(x_1) \cdots f(x_m)| \leq \varepsilon \|x_1\|^{p_1} \cdots \|x_m\|^{p_m} \tag{2}$$

for all $x_1, ..., x_m \in V$. In section 3, the following theorem shall be proved:

THEOREM 3. *If* $f(0) = 0$ *then* $f(x) = 0$ *holds for all* $x \in V$. *Otherwise, if we define* $g(x) = f(0)^{-1}f(x)$ *for all* $x \in V$ *then it holds* $g(x+y) = g(x)g(y)$ *for all* $x, y \in V$.

Moreover, the superstability problem of the functional equation $f(x_1 + \cdots + x_m) = f(x_1) \cdots f(x_m)$ shall be investigated in another setting (9).

2. Superstability of $f(x + y) = f(x)f(y)$

Let p, q, and ε be positive numbers with $p \leq q$. Suppose that V is a normed vector space and f is a complex-valued mapping defined on V which satisfies the functional inequality (1). Then, Theorem 1 in [4] says that it holds either $|f(x)| = o\left(\|x\|^{p+q}\right)$ as $\|x\| \to \infty$ or $f(x+y) = f(x)f(y)$ for all $x, y \in V$. However, precisely speaking, this theorem gives only the asymptotic behavior of the absolute value of $f(x)$ in the case when f is not exponential. Hence, it is worth while to estimate a suitable upper bound for $|f(x)|$ if f is not exponential. We shall first prove the next lemma:

LEMMA 1. *If there exists $v \neq 0$ in V satisfying $|f(v)| > \max\{2^p, \psi_{p,q}(v)\}$, then*

$$\|nv\|^q = o\left(|f(nv)|\right)$$

as $n \to \infty$.

PROOF. Using induction on n, we first prove that

$$\begin{aligned}(3) \qquad & |f(nv) - f(v)^n| \\ & \leq \varepsilon\left(|f(v)|^{n/2} + |f(v)|^{(n+1)/2} + \cdots + |f(v)|^{n-1}\right)\|v\|^{p+q}\end{aligned}$$

for each natural number n. The inequality (3) is trivial for $n = 1$. Now, assume that (3) is valid for some n. It is not difficult to verify that the condition $|f(v)| > 2^p$ implies $|f(v)|^{(n+1)/2} > n^p$ for every $n \in \mathbf{N}$. This fact, together with (1) and (3), yields

$$\begin{aligned}& |f((n+1)v) - f(v)^{n+1}| \\ & \leq |f((n+1)v) - f(nv)f(v)| + |f(nv)f(v) - f(v)^{n+1}| \\ & \leq \varepsilon\|nv\|^p\|v\|^q + |f(v)|\varepsilon\|v\|^{p+q}\left(|f(v)|^{n/2} + |f(v)|^{(n+1)/2} + \cdots + |f(v)|^{n-1}\right) \\ & \leq \varepsilon\|v\|^{p+q}\left(|f(v)|^{(n+1)/2} + |f(v)|^{(n+2)/2} + \cdots + |f(v)|^n\right).\end{aligned}$$

Hence, the inequality (3) holds for every $n \in \mathbf{N}$. If we divide both sides in (3) by $|f(v)|^n$ then the condition $|f(v)| > \psi_{p,q}(v)$ yields

$$\begin{aligned}|f(nv)f(v)^{-n} - 1| & \leq \varepsilon\|v\|^{p+q}\left(|f(v)|^{-1} + |f(v)|^{-3/2} + \cdots\right) \\ (4) \qquad & \leq \frac{\varepsilon\|v\|^{p+q}}{|f(v)| - |f(v)|^{1/2}} \\ & < 1\end{aligned}$$

for all $n \in \mathbf{N}$. On the other hand, we have

$$\frac{\|nv\|^q}{|f(v)|^n} < \frac{n^q\|v\|^q}{(1+\varepsilon\|v\|^{p+q})^n} \to 0 \text{ as } n \to \infty, \tag{5}$$

since the definition of $\psi_{p,q}(v)$ and the fact $\psi_{p,q}(v) > 1$ imply $|f(v)| > \psi_{p,q}(v) = [\psi_{p,q}(v)]^{1/2} + \varepsilon\|v\|^{p+q} > 1+\varepsilon\|v\|^{p+q}$. Comparing (4) with (5), we complete the proof of the lemma. □

Proof of Theorem 2. Assume that there exists $v \neq 0$ in V such that $|f(v)| > \max\{2^p, \psi_{p,q}(v)\}$. For any $x, y \in V$, we get

$$\begin{aligned} &|f(nv)||f(x+y) - f(x)f(y)| = |f(x+y)f(nv) - f(x)f(y)f(nv)| \\ &\leq |f(x+y)f(nv) - f(x+y+nv)| \\ &+|f(x+y+nv) - f(x)f(y+nv)| + |f(x)||f(y+nv) - f(y)f(nv)|. \end{aligned} \tag{6}$$

From (1) and (6) it follows

$$\begin{aligned} &|f(nv)||f(x+y) - f(x)f(y)| \\ &\leq \varepsilon\left(\|x+y\|^p\|nv\|^q + \|x\|^p\|y+nv\|^q + |f(x)|\|y\|^p\|nv\|^q\right) \\ &\leq C\|nv\|^q \end{aligned} \tag{7}$$

for sufficiently large n and for some suitable $C = C(x,y) > 0$. If $f(x+y) \neq f(x)f(y)$, then (7) would be contrary to Lemma 1. □

3. Superstability of $\mathbf{f(x_1 + \cdots + x_m) = f(x_1)\cdots f(x_m)}$

Suppose m (> 2) is an integer and $p_1, \ldots, p_m$ are given with $0 < p_1 \leq \cdots \leq p_m$. Assume that f is a complex-valued mapping defined on a normed vector space V and satisfying the functional inequality (2).

Proof of Theorem 3. By putting $x_1 = x_2 = \cdots = x_m = 0$ in (2), we get

$$f(0) - f(0)^m = 0. \tag{8}$$

It follows from (8) that $f(0) = 0$ or $f(0)^{m-1} = 1$.

Case I. Suppose $f(0) = 0$. Setting $x_1 = x$ and $x_2 = \cdots = x_m = 0$ in (2) yields

$$f(x) = f(x)f(0)^{m-1} = 0.$$

Case II. Assume $f(0)^{m-1} = 1$. We then have $f(0) \neq 0$ and $f(0)^{m-2} = f(0)^{-1}$. By putting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_m = 0$ in (2) it follows

$$f(x+y) = f(0)^{m-2} f(x) f(y) = f(0)^{-1} f(x) f(y).$$

Hence, if we define $g(x) = f(0)^{-1} f(x)$ for $x \in V$ then it holds $g(x+y) = g(x)g(y)$ for all $x, y \in V$. □

By using Theorem 1 in [4], we can easily prove the superstability of the functional equation $f(x_1 + \cdots + x_m) = f(x_1) \cdots f(x_m)$ in another setting as we see in the following theorem:

THEOREM 4. *Let m (> 2) be a fixed integer and let p (> 0) be given. If a complex-valued mapping f defined on V satisfies the following functional inequality*

$$(9) \qquad |f(x_1 + \cdots + x_m) - f(x_1) \cdots f(x_m)| \leq \varepsilon \left(\|x_1\|^p + \cdots + \|x_m\|^p \right)$$

for all $x_1, ..., x_m \in V$, then either $|f(x)| = O\left(\|x\|^p\right)$ as $\|x\| \to \infty$ or $g(x+y) = g(x)g(y)$ for all $x, y \in V$, where $g(x) = f(0)^{-1} f(x)$ for $x \in V$.

PROOF. Setting $x_1 = x_2 = \cdots = x_m = 0$ in (9) yields the equation (8). Therefore, it holds $f(0) = 0$ or $f(0)^{m-1} = 1$.

Case I. Let $f(0) = 0$. By putting $x_1 = x$ and $x_2 = \cdots = x_m = 0$ in (9), we have

$$|f(x)| \leq \varepsilon \|x\|^p.$$

Case II. Suppose $f(0)^{m-1} = 1$. If we define $g(x) = f(0)^{-1} f(x)$ for $x \in V$, then we obtain the functional inequality

$$|g(x+y) - g(x)g(y)| \leq \varepsilon |f(0)|^{-1} \left(\|x\|^p + \|y\|^p \right)$$

by putting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_m = 0$ in (9) and by using the fact $f(0)^{m-2} = f(0)^{-1}$. According to Theorem 1 in [4], it holds either $|f(x)| = o\left(\|x\|^p\right)$ as $\|x\| \to \infty$ or $g(x+y) = g(x)g(y)$ for all $x, y \in V$. □

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