

ON q -ANALOGUES OF STIRLING SERIES

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ABSTRACT. In this short note, we construct another form of Stirling's asymptotic series by new form of Carlitz's q -Bernoulli numbers.

0. Introduction

We use the notation

$$[x] = [x; q] = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \rightarrow 1} [x; q] = x$, for any $x \in \mathbb{C}$.

Carlitz's q -Bernoulli numbers $\beta_k = \beta_k(q)$ can be determined inductively by

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing β^i by β_i . Let u be a complex number with $|u| > 1$, then the Carlitz's q -Euler numbers $H_k(u; q)$ is defined inductively by

$$(1) \quad H_0(u; q) = 1, \quad (qH + 1)^k - uH_k(u; q) = 0$$

for $k \geq 1$, with the usual convention of replacing H^k by $H_k(u, q)$. We remark that Carlitz's q -Euler numbers are reduced to the ordinary Euler numbers $H_k(u)$ when $q = 1$.

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Let $q, u \in \mathbb{C}$ with $|q| < 1$, $|u| > 1$. For $s \in \mathbb{C}$, the complex function $l_q(s, u)$ is constructed by

$$(2) \quad l_q(s, u) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]_q^s}.$$

For any positive integer k , it can be found in [2] that

$$(3) \quad l_q(-k, u) = \begin{cases} \frac{1}{u-1}, & \text{if } k = 0, \\ \frac{u}{u-1} H_k(u, q), & \text{if } k \geq 1. \end{cases}$$

In [6], q -Riemann ζ -function was defined by

$$\zeta_q(s) = \frac{2-s}{s-1} (1-q) \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s}.$$

In this paper, we will give new relations on Carlitz's q -Bernoulli numbers and new properties of the function $G_q(x)$ in [2].

1. Relations on Carlitz's q -Bernoulli numbers

The new form of Carlitz's q -Bernoulli numbers $\mathcal{B}_k = \mathcal{B}_k(u; q)$ can be determined inductively by

$$\mathcal{B}_0(u; q) = 1, \quad u^{-1}(q\mathcal{B} + 1)^k - \mathcal{B}_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing \mathcal{B}^i by \mathcal{B}_i . The new form of Carlitz's q -Bernoulli numbers are reduced to the ordinary Carlitz's q -Bernoulli numbers $\beta_k(q)$ when $u^{-1} = q$. We shall explicitly determine the generating function $F_{u,q}(t)$ of \mathcal{B}_k :

$$F_{u,q}(t) = \sum_{k=0}^{\infty} \mathcal{B}_k(u; q) \frac{t^k}{k!}.$$

This is the unique solution of the following $(u; q)$ -difference equation

$$(4) \quad F_{u,q}(t) = u^{-1} e^t F_{u,q}(qt) + 1 - u^{-1} - t.$$

Indeed, if the power series $\sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ satisfies (4), then

$$\begin{aligned}
 & \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \\
 (5) \quad &= u^{-1} e^t \sum_{k=0}^{\infty} q^k a_k \frac{t^k}{k!} + 1 - u^{-1} - t \\
 &= u^{-1} a_0 + 1 - u^{-1} + \{u^{-1}(qa_1 + 1) - 1\}t + \sum_{k=2}^{\infty} u^{-1}(qa + 1)^k \frac{t^k}{k!},
 \end{aligned}$$

where a^i means a_i . By comparing the coefficients of both sides of (5), we see that a_k satisfies the recurrence formula of the new form of Carlitz's q -Bernoulli numbers.

LEMMA 1. For $u, q, t \in \mathbb{C}$ with $|u| > 1$, $|q| < 1$,

$$(6) \quad F_{u;q}(t) = \sum_{n=0}^{\infty} u^{-n} e^{[n]t} (1 - u^{-1} - q^n t).$$

PROOF. It is easy to show that the right side of (6) satisfies (4). \square

Note that the series on the right hand side of (6) is uniformly convergent in a wider sense. For $k \geq 0$, we have

$$\begin{aligned}
 & B_k(u; q) \\
 &= \left. \frac{d^k}{dt^k} F_{u;q}(t) \right|_{t=0} \\
 (7) \quad &= (1 - u^{-1}) \sum_{n=0}^{\infty} u^{-n} [n]^k - k \sum_{n=0}^{\infty} u^{-n} q^n [n]^{k-1} \\
 &= (1 - u^{-1}) \sum_{n=0}^{\infty} u^{-n} [n]^k + (1 - q)k \sum_{n=0}^{\infty} u^{-n} [n]^k - k \sum_{n=0}^{\infty} u^{-n} [n]^{k-1},
 \end{aligned}$$

since we used the relation $\frac{q^n}{[n]} = \frac{1}{[n]} + q - 1$.

Hence we can define the similar form of q -analogue ζ -function as follows:

DEFINITION 1. For $s \in \mathbb{C}$,

$$(8) \quad \zeta_q(u; s) = \frac{1 - u^{-1}}{s - 1} \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^{s-1}} + (q - 1) \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s}.$$

Note that $\zeta_q(u; s)$ is a meromorphic function on \mathbb{C} with only one simple pole at $s = 1$. The values of $\zeta_q(u; s)$ at non-positive integers are obtained by the following theorem.

THEOREM 1. For any positive integer k , we have

$$\zeta_q(u; 1 - k) = \begin{cases} \frac{1-q}{u-1} \mathcal{B}_1(u; q), & \text{if } k = 1, \\ -\frac{\mathcal{B}_k(u; q)}{k}, & \text{if } k > 1. \end{cases}$$

PROOF. It is clear by (7) and (8). □

If $u = q^{-1}$ then we obtain the following:

COROLLARY 1. For any positive integer k , we have

$$\zeta_q(1 - k) = \begin{cases} q\beta_1(q), & \text{if } k = 1, \\ -\frac{\beta_k(q)}{k}, & \text{if } k > 1. \end{cases}$$

2. Results

The Stirling's asymptotic series

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} \approx \left(x - \frac{1}{2}\right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} B_{k+1} \frac{1}{x^k}$$

is well known. Here, we treat similar form of q -analogue of the above formula in [2].

We have the function from [2];

$$(9) \quad G_{u; q}(x) = \sum_{n=0}^{\infty} u^{-n} (x + [n]) \log(x + [n]),$$

for $x \in \mathbb{C}$ with $|u| > 1$, $|q| < 1$. Then $G_{u; q}(x)$ is a locally analytic function on \mathbb{C} .

LEMMA 2 [2].

(1) For $x \in \mathbb{C}$ with $|x| > \frac{1}{(1-|q|)^2}$, we have

$$G_{u,q}(x) = \frac{1}{1-u^{-1}} \left(x \log x + \frac{1}{u-q} (\log x + 1) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} \right).$$

(2) Let $G'_{u,q}(x) = \frac{d}{dx} G_{u,q}(x)$. Then

$$G'_{u,q}(x) = \frac{1}{1-u^{-1}} (\log x + 1) + \frac{1}{1-u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^k}.$$

From the Theorem 1, (3) and (8), for $k \geq 2$ we have

$$\begin{aligned} & \frac{\mathcal{B}_k(u; q)}{k} \\ &= \zeta_q(u; 1-k) \\ &= -\frac{1-u^{-1}}{k} \sum_{n=1}^{\infty} u^{-n} [n]^k + (q-1) \sum_{n=1}^{\infty} u^{-n} [n]^k + \sum_{n=1}^{\infty} u^{-n} [n]^{k-1} \\ &= -\left(\frac{1-u^{-1}}{k} + (1-q) \right) \frac{1}{1-u^{-1}} H_k(u; q) + \frac{1}{1-u^{-1}} H_{k-1}(u; q) \\ &= -\left(\frac{1}{k} + \frac{1-q}{1-u^{-1}} \right) H_k(u; q) + \frac{1}{1-u^{-1}} H_{k-1}(u; q). \end{aligned}$$

Hence we obtain the following:

LEMMA 3. For integer $k \geq 2$, we have

$$-\frac{\mathcal{B}_k(u; q)}{k} = -\left(\frac{1}{k} + \frac{1-q}{1-u^{-1}} \right) H_k(u; q) + \frac{1}{1-u^{-1}} H_{k-1}(u; q).$$

Now we define the function $G_q(u; x)$ by

$$\begin{aligned} G_q(u; x) &= (qx - x - 1)G'_{u,q}(x) + (2 - q - u^{-1})G_{u,q}(x) \\ &\quad + \frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}} H_1(u; q). \end{aligned}$$

Then $G_q(u; x)$ is a locally analytic function on \mathbb{C} . If $u = q^{-1}$ then $G_q(q^{-1}, x) = G_q(x)$ which is the function $G_q(x)$ in [2].

By Lemma 2, we have

$$\begin{aligned}
& (qx - x - 1)G'_{u; q}(x) + (2 - q - u^{-1})G_{u; q}(x) \\
&= -(1 - q)xG'_{u; q}(x) - G'_{u; q}(x) + (1 - q)G_{u; q}(x) + (1 - u^{-1})G_{u; q}(x) \\
&= -(1 - q)x \left\{ \frac{1}{1 - u^{-1}}(\log x + 1) + \frac{1}{1 - u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^k} \right\} \\
&\quad - \left(\frac{1}{1 - u^{-1}}(\log x + 1) + \frac{1}{1 - u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^k} \right) \\
&\quad + (1 - q) \left\{ \frac{1}{1 - u^{-1}} \left(x \log x + \frac{1}{u - q}(\log x + 1) \right. \right. \\
&\quad \quad \quad \left. \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} \right) \right\} \\
&\quad + (1 - u^{-1}) \left\{ \frac{1}{1 - u^{-1}} \left(x \log x + \frac{1}{u - q}(\log x + 1) \right. \right. \\
&\quad \quad \quad \left. \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} \right) \right\} \\
&= -\frac{1 - q}{1 - u^{-1}} x \log x - \frac{1 - q}{1 - u^{-1}} x - \frac{1}{1 - u^{-1}} \log x - \frac{1}{1 - u^{-1}} \\
&\quad + \frac{1 - q}{1 - u^{-1}} x \log x + \frac{1 - q}{1 - u^{-1}} \frac{1}{u - q} \log x + \frac{1 - q}{1 - u^{-1}} \frac{1}{u - q} \\
&\quad + x \log x + \frac{1}{u - q} \log x + \frac{1}{u - q} \\
&\quad - \frac{1 - q}{1 - u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^{k-1}} \\
&\quad - \frac{1}{1 - u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^k} \\
&\quad + \frac{1 - q}{1 - u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k}
\end{aligned}$$

$$\begin{aligned}
 &= x \log x - \left(\frac{1}{1-u^{-1}} - \frac{1-q}{1-u^{-1}} \frac{1}{u-q} - \frac{1}{u-q} \right) \log x - \frac{1-q}{1-u^{-1}} x \\
 &\quad - \left(\frac{1}{1-u^{-1}} - \frac{1-q}{1-u^{-1}} \frac{1}{u-q} - \frac{1}{u-q} \right) - \frac{1-q}{1-u^{-1}} H_1(u; q) \\
 &\quad + \frac{1-q}{1-u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1} \frac{k}{k} H_{k+1}(u; q) \frac{1}{x^k} \\
 &\quad - \frac{1}{1-u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^k} \\
 &\quad + \frac{1-q}{1-u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} \\
 &= x \log x - \frac{1-u^{-1}}{1-qu^{-1}} \log x - \frac{1-q}{1-u^{-1}} x - \frac{1-u^{-1}}{1-qu^{-1}} - \frac{1-q}{1-u^{-1}} H_1(u; q) \\
 &\quad + \frac{1-q}{1-u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_{k+1}(u; q) \frac{1}{x^k} \\
 &\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{x^k} - \frac{1}{1-u^{-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{x^k} \\
 &= \left(x - \frac{1-u^{-1}}{1-qu^{-1}} \right) \log x - \frac{1-q}{1-u^{-1}} x - \frac{1-u^{-1}}{1-qu^{-1}} - \frac{1-q}{1-u^{-1}} H_1(u; q) \\
 &\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left\{ \frac{1-q}{1-u^{-1}} H_{k+1}(u; q) \right. \\
 &\quad \quad \left. + \frac{1}{k+1} H_{k+1}(u; q) - \frac{1}{1-u^{-1}} H_k(u; q) \right\} \frac{1}{x^k}.
 \end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}
 &G_q(u; x) \\
 &= \left(x - \frac{1-u^{-1}}{1-qu^{-1}} \right) \log x - \frac{1-q}{1-u^{-1}} x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \mathcal{B}_{k+1}(u; q) \frac{1}{x^k}.
 \end{aligned}$$

Hence we obtain the following:

THEOREM 2. For $x \in \mathbb{C}$ with $|x| > \frac{1}{(1-|q|)^2}$

$$G_q(u; x) = \left(x - \frac{1-u^{-1}}{1-qu^{-1}} \right) \log x - \frac{1-q}{1-u^{-1}} x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \mathcal{B}_{k+1}(u; q) \frac{1}{x^k}.$$

If $u = q^{-1}$, then we obtain the following:

COROLLARY 1 [2]. For $x \in \mathbb{C}$ with $|x| > \frac{1}{(1-|q|)^2}$

$$G_q(x) = \left(x - \frac{1}{[2]} \right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \beta_{k+1}(q) \frac{1}{x^k}.$$

Now, we give another result.

We define the function $\mathcal{G}_{u,q}(x)$ which is the q -analogue of x in (9):

$$\mathcal{G}_{u,q}(x) = \sum_{n=0}^{\infty} u^{-n}([x] + [n]) \log([x] + [n]),$$

for $x \in \mathbb{C}$, with $|u| > 1$, $|q| < 1$. Then $\mathcal{G}_{u,q}(x)$ is a locally analytic function on \mathbb{C} .

For $|[x]| > \frac{1}{(1-|q|)^2}$, we have

$$\begin{aligned} \mathcal{G}_{u,q}(x) &= \sum_{n=1}^{\infty} u^{-n}([x] + [n]) \log([x] + [n]) \\ &= \sum_{n=1}^{\infty} u^{-n}([x] + [n]) \log[x] + \sum_{n=1}^{\infty} u^{-n}([x] + [n]) \log \left(1 + \frac{[n]}{[x]} \right) + [x] \log[x] \\ &= \left(\sum_{n=1}^{\infty} u^{-n} \right) [x] \log[x] + \left(\sum_{n=1}^{\infty} u^{-n} [n] \right) \log[x] \\ &\quad + \sum_{n=1}^{\infty} u^{-n} \left([n] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \frac{[n]^{k+1}}{[x]^k} \right) + [x] \log[x] \\ &= \frac{1}{u-1} [x] \log[x] + \frac{u}{u-1} H_1(u; q) \log[x] + \frac{u}{u-1} H_1(u; q) \\ &\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \frac{u}{u-1} H_{k+1}(u; q) \frac{1}{[x]^k} + [x] \log[x]. \end{aligned}$$

Since $H_1(u; q) = \frac{1}{u-q}$, we have

$$\begin{aligned} \mathcal{G}_{u; q}(x) &= \frac{u}{u-1} [x] \log [x] + \frac{u}{u-1} \left(\frac{1}{u-q} (1 + \log [x]) \right) \\ &\quad + \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{[x]^k} \\ &= \frac{u}{u-1} \left\{ [x] \log [x] + \frac{1}{u-q} (1 + \log [x]) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{[x]^k} \right\}. \end{aligned}$$

Let $\mathcal{G}'_{u; q}(x) = \frac{d}{dx} \mathcal{G}_{u; q}(x)$, then we have

$$\begin{aligned} \mathcal{G}'_{u; q}(x) &= \frac{u}{u-1} \left\{ -\frac{q^x}{1-q} \log q \log [x] + \left(-\frac{q^x}{1-q^x} \log q \right) [x] \right. \\ &\quad \left. + \frac{1}{u-q} \left(-\frac{q^x}{1-q^x} \log q \right) \right\} \\ &\quad + \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \left(\frac{q^x}{1-q} k \log q \frac{1}{[x]^{k+1}} \right) \\ &= -\frac{u}{u-1} \frac{q^x}{1-q} \log q (\log [x] + 1) - \frac{u}{u-1} \frac{q^x}{1-q} \log q H_1(u; q) \frac{1}{[x]} \\ &\quad - \frac{u}{u-1} \frac{q^x}{1-q} \log q \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{[x]^k} \\ &= -\frac{u}{u-1} \frac{q^x}{1-q} \log q (\log [x] + 1) \\ &\quad - \frac{u}{u-1} \frac{q^x}{1-q} \log q \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{[x]^k}. \end{aligned}$$

Therefore we obtain the following:

PROPOSITION 1.

(1) For $x \in \mathbb{C}$ with $||[x]|| > \frac{1}{(1-|q|)^2}$, we have

$$\mathcal{G}_{u;q}(x) = \frac{1}{1-u^{-1}} \left\{ [x] \log[x] + \frac{1}{u-q} (\log[x] + 1) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(u; q) \frac{1}{[x]^k} \right\}.$$

(2)

$$\begin{aligned} & \mathcal{G}'_{u;q}(x) \\ &= -\frac{1}{1-u^{-1}} \frac{q^x}{1-q} \log q \left\{ (\log[x] + 1) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u; q) \frac{1}{[x]^k} \right\}. \end{aligned}$$

Now we rewrite the function $\mathcal{G}_q(u; x)$ by

$$\begin{aligned} \mathcal{G}_q(u; x) &= (q[x] - [x] - 1) \mathcal{G}'_{u;q}(x) - \frac{q^x}{1-q} \log q (2 - q - u^{-1}) \mathcal{G}_{u;q}(x) \\ &\quad - \frac{q^x}{1-q} \log q \left(\frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}} H_1(u; q) \right). \end{aligned}$$

Then $\mathcal{G}_q(u; x)$ is a locally analytic function on \mathbb{C} . By Theorem 2, we have

$$\begin{aligned} & \mathcal{G}_q(u; x) \\ &= (q[x] - [x] - 1) \mathcal{G}'_{u;q}(x) - \frac{q^x}{1-q} \log q (2 - q - u^{-1}) \mathcal{G}_{u;q}(x) \\ &\quad - \frac{q^x}{1-q} \log q \left(\frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}} H_1(u; q) \right) \\ &= -\frac{q^x}{1-q} \log q \left\{ (q[x] - [x] - 1) \mathcal{G}'_{u;q}([x]) + (2 - q - u^{-1}) \mathcal{G}_{u;q}([x]) \right. \\ &\quad \left. + \frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}} H_1(u; q) \right\} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{q^x}{1-q} \log q G_q(u; [x]) \\
 &= -\frac{q^x}{1-q} \log q \left\{ \left([x] + \frac{1-u^{-1}}{1-qu^{-1}} \right) \log [x] - \frac{1-q}{1-u^{-1}} [x] \right. \\
 &\quad \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \mathcal{B}_{k+1}(u; q) \frac{1}{[x]^k} \right\}.
 \end{aligned}$$

Hence we obtain the following:

THEOREM 3. For $x \in \mathbb{C}$ with $|[x]| > \frac{1}{(1-|q|)^2}$ we have

$$\begin{aligned}
 \mathcal{G}_q(u; x) = &-\frac{q^x}{1-q} \log q \left\{ \left([x] + \frac{1-u^{-1}}{1-qu^{-1}} \right) \log [x] - \frac{1-q}{1-u^{-1}} [x] \right. \\
 &\left. + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \mathcal{B}_{k+1}(u; q) \frac{1}{[x]^k} \right\}.
 \end{aligned}$$

Note that if $u = q^{-1}$ and $q \rightarrow 1$, then we have

$$\begin{aligned}
 &\lim_{q \rightarrow 1} \mathcal{G}_q(q^{-1}, x) \\
 &= \lim_{q \rightarrow 1} \left\{ -\frac{q^x}{1-q} \log q \left(\left([x] + \frac{1}{[2]} \right) \log [x] - [x] \right. \right. \\
 &\quad \left. \left. + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \beta_{k+1}(q) \frac{1}{[x]^k} \right) \right\} \\
 &= \left(x - \frac{1}{2} \right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} B_{k+1} \frac{1}{x^k},
 \end{aligned}$$

which is the well known Stirling asymptotic series.

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