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ASYMPTOTIC FUNCTIONS

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Abstract. In this paper, we improve some of results in [2] by showing that if I is a cancellation ideal and if J is a regular ideal then $\alpha(m)$, $\beta(m)$ and $\delta(m)$, behave nicely under localization. We prove that $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$ if and only if $\alpha(m)$ is eventually constant and that $\lim_{n\to\infty} \frac{\alpha(n)}{n}$ exists and is equal to or less than $\alpha(1)$. Finally we give several conditions which are equivalent to $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$.

1. Introduction

Throughout this paper, R will always be a commutative Noetherian ring and I and J will be ideals in R unless otherwise stated. If $J \subseteq I$ then we denote by $(J : I) = \{r \in R : rI \subseteq J\}$ the annihilator of I/J. For each $n \geq 1$, since $(I^n : J) \subseteq (I^n : J^2) \subseteq (I^n : J^3) \subseteq \cdots$ is an increasing sequence, the sequence eventually stabilizes, that is, $(I^n : J^k) = (I^n : J^{k+1}) = \cdots$ for all large integer k. We define $\alpha(n)$ to be the least such k. If J is regular then, for each $m \geq 1$, it is not hard to show that there is an integer h such that $(I^{h+r} : J^m) \subseteq I^r$, for all $r \geq 1$. We define $\beta(m)$ to be the least such h.

In [2], Katz and McAdam introduced these two functions: $\alpha(m)$ and $\beta(m)$, and studied the behavior of $\frac{\alpha(m)}{m}$ as $m \to \infty$ and $\frac{\beta(m)}{m}$ as $m \to \infty$. They showed that if J is a regular ideal, then $\lim_{m\to\infty} \frac{\beta(m)}{m}$ exists and that $\lim_{m\to\infty} \frac{\beta(m)}{m} = 0$ if and only if $\{\beta(m) : m \geq 1\}$ is eventually constant. They also showed that if I is a regular principal ideal then $\lim_{m\to\infty} \frac{\alpha(m)}{m}$ exists and

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is equal to or less than $\alpha(1)$. Furthermore, if J is also a regular ideal, then $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$ if and only if $\{\alpha(m) : m \geq 1\}$ is eventually constant.

The purpose of the present paper is to improve this result by replacing a regular principal ideal with a cancellation ideal.

We first prove in theorem (2.5) that if I is a cancellation ideal then

$$Ass_R(R/I) = Ass_R(R/I^n), \text{ for all } n \geq 1.$$

In lemma (3.1), we show that the asymptotic functions: $\alpha(m)$, $\beta(m)$ and $\delta(m)$, behave nicely under localization. This allows us to prove in theorem (3.6) that if I is a cancellation ideal of R and J is a regular ideal, then $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$ if and only if $\alpha(m)$ is eventually constant. We also prove in theorem (3.3) that if I is a cancellation ideal of R, then $\lim_{n\to\infty} \frac{\alpha(n)}{n}$ exists and is equal to or less than $\alpha(1)$. Finally we show in corollary (3.7) that the conditions in corollary (3.7) are equivalent.

2. Definitions and preliminary results

We begin this section by listing the definitions and some results that will be needed in this paper.

Definition 2.1. Let R be a ring with identity. A nonzero ideal I of R is said to be a cancellation ideal if I has the property that, for any ideals J, K of R, $IJ \subseteq IK$ implies that $J \subseteq K$.

The following results concerning cancellation ideals are well known. For a proof, we refer the reader to [1].

Remark 2.2. Let R be a commutative ring with identity, $I, I_1, I_2, \cdots, I_{t-1}$ and I_t ideals of R with I cancellative and S a multiplicative subset of R. Then

(1) $I_1 I_2 \cdots I_t$ is a cancellation ideal of R if and only if each I_i is a cancellation ideal of R. In particular, I^n is a cancellation ideal, for all $n \geq 1$.
(2) IR_S is a cancellation ideal of R_S.
(3) If R is a quasi-local ring and if I is finitely generated then I is a regular principal ideal.

DEFINITION 2.3. Let R be a Noetherian ring, I and J ideals of R with J regular and m a positive integer.

(A) $\alpha(m)$ is the least positive integer k such that $(I^m : J^k) = (I^m : J^{k+r})$ for all integer $r \geq 1$.
(B) $\beta(m)$ is the least positive integer h such that $(I^{h+r} : J^m) \subseteq I^r$, for all integer $r \geq 1$.
(C) $\delta(m)$ is the least positive integer t such that $(I^t : J^m) \subseteq I$.

The existence of $\alpha(m)$ was shown in the introduction and the existence of $\beta(m)$ was shown in [2]. By the definition, $1 \leq \delta(m) \leq \beta(m) + 1$ and since $(I^{\delta(m+1)} : J^m) \subseteq (I^{\delta(m+1)} : J^{m+1}) \subseteq I$, $\delta(m) \leq \delta(m+1)$ for all $m \geq 1$, i.e., $\{\delta(m) : m \geq 1\}$ is a nondecreasing sequence. Some preliminary results which are proved by Katz and McAdam in [2] are listed in the next remark. We will improve (7), (8), (9), (10) and (11) by replacing a regular principal ideal with a cancellation ideal.

REMARK 2.4. Let R be a Noetherian ring and I and J ideals of R with J regular and let $A = \limsup\{\frac{\alpha(m)}{m} : m \geq 1\}$ and $D = \limsup\{\frac{\delta(m)}{m} : m \geq 1\}$. Then the following hold.

(1) $\{\alpha(m) : m \geq 1\}$ is eventually nondecreasing.
(2) $\{\beta(m) : m \geq 1\}$ is nondecreasing.
(3) $\{\alpha(m) : m \geq 1\}$ is eventually constant if and only if so is $\{\beta(m) : m \geq 1\}$.
(4) $\lim_{m\to\infty} \frac{\beta(m)}{m} = 0$ if and only if $\{\beta(m) : m \geq 1\}$ is eventually constant.
(5) $\lim_{m\to\infty} \frac{\delta(m)}{m} = 0$ if and only if $\{\delta(m) : m \geq 1\}$ is eventually constant.
(6) either $AD \geq 1$ or $\{\delta(m) : m \geq 1\}$ is eventually constant.

Furthermore if I is a regular principal ideal, then

(7) $\lim_{m\to\infty} \frac{\alpha(m)}{m}$ exists and is equal to or less than $\alpha(1)$.
(8) $\delta(m) = \beta(m) + 1$, for all $m \geq 1$.
(9) $\{\beta(m) : m \geq 1\}$ is eventually constant if and only if $\{\delta(m) : m \geq 1\}$ is eventually constant.
(10) $\alpha(m+n) \leq \alpha(m) + \alpha(n)$, for all $m,\ n \geq 1$.
(11) $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$ if and only if $\{\alpha(m) : m \geq 1\}$ is eventually constant.

It is well known that if I is a regular principal ideal then $Ass_R(R/I) = Ass_R(R/I^n)$, for all $n \geq 1$ and that if S is a multiplicative subset of R and if M is a finitely generated R-module then

$$Ass_{R_S}(M_S) = Ass_R(M) \bigcap Spec(R_S).$$

Now we are ready to show theorem (2.5) that is one of our main tools in this paper.

THEOREM 2.5. *Let R be a Noetherian ring and I a cancellation ideal in R. Then $Ass_R(R/I) = Ass_R(R/I^n)$ for all $n \geq 1$.*

PROOF. For each $P \in Spec(R)$, since IR_P is a cancellation ideal in R_P, IR_P is a regular principal ideal. Hence

$$Ass_{R_P}(R_P/IR_P) = Ass_{R_P}(R_P/(I^n)R_P), \quad \text{for all } n \geq 1.$$

If $P \in Ass_R(R/I)$ then

$$P \in Ass_R(R/I) \bigcap Spec(R_P) = Ass_{R_P}(R_P/IR_P).$$

Since $Ass_{R_P}(R_P/IR_P) = Ass_{R_P}(R_P/(I^n)R_P)$,

$$P \in Ass_{R_P}(R_P/(I^n)R_P) = Ass_R(R/I^n) \bigcap Spec(R_P).$$

Thus $P \in Ass_R(R/I^n)$, i.e., $Ass_R(R/I) \subseteq Ass_R(R/I^n)$, for all $n \geq 1$. Similarly, the opposite inclusion holds. Hence the theorem follows. □

3. Main results

In this section, we will prove theorem (3.3) and theorem (3.6) which are our main theorems in this paper. To give a proof of main theorems, we first prove lemma (3.1) that shows that asymptotic functions: $\alpha(m), \beta(m)$ and $\delta(m)$, behave nicely under localization. Throughout this paper, we will denote by $\alpha_P(m)$, $\beta_P(m)$ and $\delta_P(m)$ the asymptotic functions with respect to IR_P and JR_P, respectively.

LEMMA 3.1. *Let R be a Noetherian ring and I a cancellation ideal in R. Then the following hold.*

(1) $\alpha(m) = \max\{\alpha_P(m) : P \in Ass_R(R/I)\}$ *for all* $m \geq 1$.
(2) $\beta(m) = \max\{\beta_P(m) : P \in Ass_R(R/I)\}$ *for all* $m \geq 1$.
(3) $\delta(m) = \max\{\delta_P(m) : P \in Ass_R(R/I)\}$ *for all* $m \geq 1$.

PROOF. For each $m \geq 1$, $(I^m : J^{\alpha(m)}) = (I^m : J^{\alpha(m)+k})$, for all $k \geq 0$. Thus for each $P \in Spec(R)$, $(I^m R_P : J^{\alpha(m)} R_P) = (I^m R_P : J^{\alpha(m)+k} R_P)$, for all $k \geq 0$. The minimality of $\alpha_P(m)$ shows that $\alpha_P(m) \leq \alpha(m)$ for any $P \in Spec(R)$. Thus set $h(m) = \max\{\alpha_P(m) : P \in Ass_R(R/I)\}$, then $h(m) \leq \alpha(m)$. Conversely, the maximality of $h(m)$ shows that for all $P \in Ass_R(R/I)$ and $k \geq 0$, $(I^m R_P : J^{h(m)} R_P) = (I^m R_P : J^{h(m)+k} R_P)$. Since $Ass_R(R/I) = Ass_R(R/I^n)$ for all $n \geq 1$, it is not hard to see that $(I^m : J^{h(m)}) = (I^m : J^{h(m)+k})$. Thus by the definition of $\alpha(m)$, $\alpha(m) \leq h(m)$. Hence $\alpha(m) = \max\{\alpha_P(m) : P \in Ass_R(R/I)\}$ for all $m \geq 1$.

For (2), it suffices to show that $\beta(m) \leq \beta_P(m)$, for any $P \in Ass_R(R/I)$. Let $g(m) = \max\{\beta_P(m) : P \in Ass_R(R/I)\}$. Then by the maximality of $g(m)$, for all $P \in Ass_R(R/I)$ and $r \geq 1$, $(I^{g(m)+r} R_P : J^m R_P) \subseteq I^r R_P$. Thus

$$(I^{g(m)+r} : J^m) \subseteq (I^{g(m)+r} R_P : J^m R_P) \bigcap R \subseteq I^r R_P \bigcap R,$$

for all $P \in Ass_R(R/I)$. Thus $(I^{g(m)+r} : J^m) \subseteq \bigcap_{P \in Ass_R(R/I)} (I^r R_P \bigcap R) = I^r$. The minimality of $\beta(m)$ shows that $\beta(m) \leq g(m)$. (3) follows from (2). □

LEMMA 3.2. *Let R be a Noetherian ring and I a cancellation ideal in R. Then $\alpha(m+n) \leq \alpha(m) + \alpha(n)$.*

PROOF. For each $P \in Spec(R)$, since IR_P is a regular principal ideal in R_P, $\alpha_P(m+n) \leq \alpha_P(m) + \alpha_P(n)$, for all $m,\ n \geq 1$. Thus

$$\begin{aligned}
\alpha(m+n) &= \max\{\alpha_P(m+n) : P \in Ass_R(R/I)\} \\
&\leq \max\{\alpha_P(m) + \alpha_P(n) : P \in Ass_R(R/I)\} \\
&\leq \max\{\alpha_P(m) : P \in Ass_R(R/I)\} \\
&+ \max\{\alpha_P(n) : P \in Ass_R(R/I)\} \\
&= \alpha(m) + \alpha(n).
\end{aligned}$$

□

Lemma (3.2) allows us to use the same argument used in [2]. The proof in [2] is easily carried over to $\alpha(m)$ so that we omit the proof. For a proof, we refer to [2, Proposition (1.3)].

THEOREM 3.3. *Let R be a Noetherian ring and I a cancellation ideal in R. Then $\lim_{m\to\infty} \frac{\alpha(m)}{m}$ exists and is equal to or less than $\alpha(1)$.*

PROPOSITION 3.4. *Let R be a Noetherian ring and I a cancellation ideal in R. If J is a regular ideal then $\delta(m) = \beta(m) + 1$ for all $m \geq 1$.*

PROOF. For each $P \in Ass_R(R/I)$, since IR_P is a regular principal ideal of R_P, $\delta_P(m) = \beta_P(m)+1$ for all $m \geq 1$. Thus there exists $P \in Ass_R(R/I)$ such that $\beta(m) = \beta_P(m)$ and $\delta(m) = \delta_P(m)$. Hence the proposition follows. □

COROLLARY 3.5. *Let R be a Noetherian ring, I a cancellation ideal in R and J a regular ideal of R. Then $\{\beta(m) : m \geq 1\}$ is eventually constant if and only if $\{\delta(m) : m \geq 1\}$ is eventually constant.*

PROOF. By proposition (3.4), $\delta(m) = \beta(m) + 1$ for all $m \geq 1$. Hence $\{\beta(m) : m \geq 1\}$ is eventually constant if and only if so is $\{\delta(m) : m \geq 1\}$. □

THEOREM 3.6. *Let R be a Noetherian ring, I a cancellation ideal in R and J a regular ideal. Then $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$ if and only if $\alpha(m)$ is eventually constant.*

PROOF. One direction of the equivalence is trivial. If $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$ then by (6) in remark (2.4), $\{\delta(m) : m \geq 1\}$ is eventually constant. By corollary (3.5), $\{\beta(m) : m \geq 1\}$ is eventually constant. By (3) in remark (2.4), $\{\alpha(m) : m \geq 1\}$ is eventually constant. This completes the proof. □

We now close this section by stating what we have shown in this paper.

COROLLARY 3.7. *Let R be a Noetherian ring, I a cancellation ideal in R and J a regular ideal of R. Then the following are equivalent.*

(1) $\lim_{m\to\infty} \frac{\alpha(m)}{m} = 0$.
(2) $\lim_{m\to\infty} \frac{\beta(m)}{m} = 0$.
(3) $\lim_{m\to\infty} \frac{\delta(m)}{m} = 0$.
(4) $\{\alpha(m) : m \geq 1\}$ *is eventually constant.*
(5) $\{\beta(m) : m \geq 1\}$ *is eventually constant.*
(6) $\{\delta(m) : m \geq 1\}$ *is eventually constant.*

PROOF. This follows from (3), (4) and (5) in remark (2.4), corollary (3.5) and theorem (3.6). □

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