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ON THE CHAIN CONDITIONS OF A FAITHFUL ENDO-FLAT MODULE

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ABSTRACT. The *faithful* bi-module ${}_RM_{End_R(M)}$ with its endomorphism ring $End_R(M)$ such that $M_{End_R(M)}$ is *flat* (in other words, $End_R(M)$*–flat*, or *endo-flat*) and with a commutative ring R containing an identity has been studied in this paper. The chain conditions of a *faithful endo-flat* module ${}_RM$ relative to those of the endomorphism ring $End_R(M)$ having the zero annihilator of each non-zero endomorphism are studied.

1. Introduction

The chain conditions of the endomorphism ring $End_R(M)$ and those of a *flat* module ${}_RM$ ([6]) were studied when ${}_RM$ is *closedly quotient endo-flat.* A left R–module ${}_RM$ is said to be *closedly quotient endo-flat* if for each *closed* submodule $N \leq M$, the quotient module M/N is *endo-flat.* Here, one of replacements of *closedly quotient endo-flat*ness which can give clues to relationships between the chain conditions of the endomorphism ring and those of a module is found. In other words, one of the replacements of *closedly quotient endo-flat*ness is that $End_R(M)$ has the zero left annihilator of each non-zero endomorphism.

The author investigates again the tools (used in [6]), a left ideal

$$I^L = Hom_R(M, L) = \{ f \in End_R(M) \mid Imf \leq L \} \trianglelefteq_l End_R(M)$$

of $End_R(M)$ and a right ideal

$$I_N = \{ f \in End_R(M) \mid N \leq \ker f \} \trianglelefteq_r End_R(M)$$

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of $End_R(M)$ for each submodules L , $N \leq M$. For a left (or right, or two-sided) ideal J of $End_R(M)$, we define the image and the kernel of J by

$$ImJ = \sum_{f \in J} Imf \ = MJ \quad \text{and} \quad \ker J = \bigcap_{f \in J} \ker f \ , \ \text{respectively} .$$

Assume that R is a commutaive ring with an identity. Because we are studying ${}_R M_{End_R(M)}$, the compositions of mappings will follow the direction of arrows:

$$fg \ : \ A \xrightarrow{\ f \ } B \xrightarrow{\ g \ } C \ .$$

The following definition is one of the equivalent definitions of an *endo-flat* (or $End_R(M)-$*flat*) module ([1],[2],[5]).

DEFINITION 1.1. A left $R-$module ${}_R M$ with its endomorphism ring $End_R(M)$, denoted by S briefly, is said to be *endo-flat* ($S-$*flat*, or *flat* over S) provided that for any left ideal J of S, there exists a $\mathbb{Z}-$isomorphism $\mu_J \ : \ M \otimes_S J \ \to \ MJ$ where μ_J is the restriction of μ to $M \otimes_S J$ for $\mu \ : M \otimes_S S \to M$ defined by $(m \otimes f)\mu \ = \ mf$ for all $m \in M$ and for all $f \in S$ and where $\mathbb{Z}$ is the ring of integers. Note that we have the following commutative diagram:

$$\begin{array}{ccc} M \otimes_S J & \xrightarrow{1_M \otimes \iota} & M \otimes_S S \\ {\scriptstyle \mu_J} \downarrow & & \downarrow {\scriptstyle \mu} \\ MJ & \xrightarrow{\ \hookrightarrow \ } & MS = M \ , \end{array}$$

with the inclusion mappings $\hookrightarrow$, ι, and the identity mapping 1_M on M.

For a commutative ring R, the abelian group ${}_R M \otimes_S S$ is an $R-$module.

Now recall that a submodule K is said to be *open* if $K = K^o$ where $K^o = \cap\{N_\alpha \leq M \mid I^{N_\alpha} = I^K\}$ and a submodule N is said to be *closed* if $N = \overline{N}$ where $\overline{N} \ = \ \sum_\alpha \{ \ N_\alpha \mid I_{N_\alpha} \ = \ I_N \ \}$.

A submodule $K \leq M$ is said to be *generated* by M if $K = \sum_{\alpha \in A} Mf_\alpha$ $= \sum_{\alpha \in A} Imf_\alpha$, for some endomorphism f_α and for some index set A.

The following lemma and theorem are well-known.

LEMMA 1.2 ([3]). *A faithful module $_RU$ is flat over its endomorphism ring, if and only if, it generates the kernel of each homomorphism*

$$d : U^{(n)} \rightarrow U \ (n = 1, 2, 3, \cdots) ,$$

where $U^{(n)}$ denotes the direct product of $n-$copies of U.

THEOREM 1.3 ([5]). *A module M_S is flat if and only if for every relation*

$$\sum_{j=1}^{n} v_j g_j = 0 \ \ (v_j \in M, \ g_j \in S)$$

there exist elements $u_1, \cdots, u_m \in M$ and elements $f_{ij} \in S$ $(i = 1, \cdots, m, \ j = 1, \cdots, n)$ such that

$$\sum_{i=1}^{m} u_i f_{ij} = v_j \ (j = 1, \cdots, n) \ \text{ and } \ \sum_{j=1}^{n} f_{ij} g_j = 0 \ \ (i = 1, \cdots, m) .$$

For any commutative ring R, we have the $R-$isomorphism $\mu : M \otimes_S S \rightarrow M$ defined by $(m \otimes f)\mu = mf$ for every $m \in M$ and every $f \in S$.

If two left (or right, or two-sided) ideals J and J' of S have the same image, then we will call J and J' *similar.* And if their kernels are identical, then we will call J and J' *cosimilar.* Furthermore *similarity* and *cosimilarity* on the lattice of all submodules are equivalence relations. $_{sim}\!\sim$ denotes the "*similarity*" and $_{cosim}\!\simeq$ denotes the "*cosimilarity*". A submodule $N \leq {}_RM$ is said to be *fully invariant* if $Nf \leq N$ for any endomorphism f.

We notice that for any left ideal $J \trianglelefteq_l S$, $\ker J = \bigcap_{f \in J} \ker f$ is always a *closed fully invariant* submodule of M, and for any right ideal $J \trianglelefteq_r S$, ImJ is an *open fully invariant* submodule of M.

2. Relationships between similarity and cosimilarity

In this section we only consider the relationships between the *similarity* and the *cosimilarity* of left ideals of the endomorphism ring S in which the left annihilator of each non-zero endomorphism is the zero.

For a *fully invariant* submodule $N \leq M$, M/N is a right $S-$module and $M/N \otimes_S J$ is a left $R-$module for any left ideal $J \trianglelefteq_l S$. And for any left ideal $J \trianglelefteq_l S$, $M/\ker J \otimes_S J$ is well-defined and is a left $R-$module because $\ker J$ is a *closed fully invariant* submodule of M.

For an *endo-flat* module M, we define a mapping

$$\rho_J : M/\ker J \otimes_S J \rightarrow MJ$$

by $((m+\ker J)\otimes f)\rho_J = mf$ for every $((m+\ker J)\otimes f) \in M/\ker J \otimes_S J$. Then ρ_J is an $R-$isomorphism.

Let $\pi_J : M \rightarrow M/\ker J$ be the natural (canonical) projection defined by $m\pi_J = m+\ker J$, for each $m \in M$ and let $1 : S \rightarrow S$ be the identity function. Define

$$\pi_J \otimes 1 : M \otimes_S J \rightarrow M/\ker J \otimes_S J$$

the tensor product of π_J and 1. Then

$$\pi_J \otimes 1 = \mu_J \rho_J^{-1} : M \otimes_S J \xrightarrow{\mu_J} MJ \xrightarrow{\rho_J^{-1}} M/\ker J \otimes_S J$$

is an $R-$isomorphism.

REMARK 2.1. For any *faithful endo-flat* module $_RM$ if every non-zero endomorphism g in $End_R(M)$ has the zero left annihilator, i.e.,

$$Ann_l(g) = \{ f \in End_R(M) \mid fg = 0 \} = 0 ,$$

then it follows that $MJ \cap \ker J = 0 = MI^{MJ} \cap \ker J$ for every left ideal $J \trianglelefteq_l End_R(M)$ which are obtained immediately from Theorem 1.3.

Therefore we obtain the following theorem.

THEOREM 2.2. *If a left $R-$faithful module $_RM$ is endo-flat and if its endomorphism ring $End_R(M)$ has the zero left annihilator of each non-zero endomorphism, then we have the following:*

For left ideals J, J' of $End_R(M)$, if J and J' are similar, then J and J' are cosimilar.

PROOF. Let S denote the endomorphism ring $End_R(M)$ of ${}_RM$. Since the left ideals J, J' are *similar* and since J, I^{MJ} are also *similar*, it suffices to show that J and I^{MJ} are *cosimilar* because if once it were proven then the fact $I^{MJ} = I^{MJ'}$ would induce *cosimilarity* of *J and J'*. Since $\ker J$, $\ker I^{MJ}$ are *fully invariant*, the tensor products

$M/\ker I^{MJ} \otimes_S J$, $M/\ker J \otimes_S J$, $M/\ker J \otimes_S I^{MJ}$, and $M/\ker I^{MJ} \otimes_S I^{MJ}$

are well-defined and they are $R-$modules. Since $J \subseteq I^{MJ}$, $\ker J \supseteq \ker I^{MJ}$ follows.

Let's consider the mappings

$$j \ : M/\ker I^{MJ} \to M/\ker J$$

defined by $(m + \ker I^{MJ})j = m + \ker J$, for every element $m + \ker I^{MJ}$ in $M/\ker I^{MJ}$,

$$\rho_{_{IMJ}} \ : M/\ker I^{MJ} \otimes_S I^{MJ} \to MI^{MJ} = MJ$$

defined by

$$((m + \ker I^{MJ}) \otimes f)\rho_{_{IMJ}} = mf$$

for every $(m + \ker I^{MJ}) \otimes f \in M/\ker I^{MJ} \otimes_S I^{MJ}$,

and

$\rho_{_J} \ : M/\ker J \otimes_S J \to MJ$ defined by $((m + \ker J) \otimes h)\rho_{_J} = mh$,

for every $(m + \ker J) \otimes h \in M/\ker J \otimes_S J$. In fact, $\rho_{_{IMJ}}$ and $\rho_{_J}$ are $R-$isomorphisms.

We can consider the following diagrams (1*) and (2*) in which mappings j , $\pi_{_J}$, $\pi_{_{IMJ}}$, $\mu_{_J}$, $\mu_{_{IMJ}}$, $\rho_{_J}$, and $\rho_{_{IMJ}}$ are involved.

$$\begin{array}{ccccccc}
M \otimes_S J & \overset{\pi_{_{IMJ}} \otimes 1_J}{\to} & M/\ker I^{MJ} \otimes_S J & \overset{1 \otimes \iota}{\to} & M/\ker I^{MJ} \otimes_S I^{MJ} & & \\
 & & j \otimes 1_J \swarrow & & j \otimes 1 \swarrow & & \searrow (\pi_{_{IMJ}} \otimes 1)^{-1} \\
\mu_J \searrow & & M/\ker J \otimes_S J & \overset{1_{M/\ker J} \otimes \iota}{\longrightarrow} & M/\ker J \otimes_S I^{MJ} & \downarrow \rho_{_{IMJ}} & M \otimes_S I^{MJ} \\
 & & \downarrow \rho_J & & & & \swarrow \mu_{_{IMJ}} \\
 & & MJ & = & & MI^{MJ} &
\end{array}$$

(1*)

Since $(\pi_{_{IMJ}} \otimes 1_J)(1 \otimes \iota)\rho_{_{IMJ}} = \mu_{_J}$ which is an $R-$isomorphism, $\pi_{_{IMJ}} \otimes 1_J$ is an $R-$monomorphism and thus $\pi_{_{IMJ}} \otimes 1_J$ is an isomorphism. Because of the facts that

$$j \otimes 1_J = (\pi_{_{IMJ}} \otimes 1_J)^{-1} \mu_{_J} \rho_{_J}{}^{-1}$$

and that

$$1 \otimes \iota = (\pi_{_{IMJ}} \otimes 1_J)^{-1}\ \mu_{_J} \mu_{_{IMJ}}{}^{-1}\ (\pi_{_{IMJ}} \otimes 1_J)\ = (\pi_{_{IMJ}} \otimes 1_J)^{-1}\ \mu_{_J}\ \rho_{_{IMJ}},$$

it follows that

$$j \otimes 1_J\ : M/\ker I^{MJ} \otimes_S J \to M/\ker J \otimes_S J$$

with the identity mapping $1_J : J \to J$ and

$$1 \otimes \iota\ : M/\ker I^{MJ} \otimes_S J \to M/\ker I^{MJ} \otimes_S I^{MJ}$$

are $R-$isomorphisms also. In the diagram (2*):

$$\begin{array}{ccccccc}
 & & M/\ker I^{MJ} \otimes_S I^{MJ} & & & & \\
 & & & \searrow^{j\otimes 1} & & & \\
\rho_{_{IMJ}} \swarrow & & M/\ker J \otimes_S J & & M/\ker J \otimes_S I^{MJ} & \overset{\otimes}{\leftarrow} & M/\ker J \times I^{MJ} \\
 & \rho_{_J} \swarrow & \phi \uparrow\ \downarrow \zeta & & \downarrow \eta & & \beta \swarrow \\
MI^{MJ} = MJ & & MJ/(\ker J \cap MJ) & = & MI^{MJ}/(\ker J \cap MI^{MJ}) & & \\
 & & \| & & & & \\
 & & MJ/0 & & & &
\end{array}$$

(2*)

for an $S-$balanced mapping

$$\beta\ : M/\ker J \times I^{MJ} \to MI^{MJ}/(\ker J \cap MI^{MJ}) \quad \text{defined by}$$

$(m+ker J, g)\beta = mg + \ker J \cap MI^{MJ}$ for every element $(m + \ker J, g)$ in $M/\ker J \times I^{MJ}$, there is a unique

$R-$ homomorphism $\eta \ : M/\ker J \otimes_S I^{MJ} \to MI^{MJ}/(\ker J \cap MI^{MJ})$

such that $\otimes\eta = \beta$. By Remark 2.1, for any element

$$\sum_1^k m_i f_i + \ker J \cap MJ = \sum_1^k m_i f_i + 0 \in MJ/(\ker J \cap MJ) = MJ/0\ ,$$

we can define

$$\phi \ : \ MJ/0 \to M/\ker J \otimes_S J \ \text{ by}$$

$$\left(\sum_1^k m_i f_i + 0\right)\phi = \sum_1^k (m_i + \ker J) \otimes f_i\ ,$$

from which, we have that

$$\phi \ : MJ/(\ker J \cap MJ) = MJ/0 \to M/\ker J \otimes_S J$$

is clearly well-defined and is an $R-$monomorphism since the restriction of β to $M/\ker J \times J$ induces a unique $R-$homomorphism

$$\zeta \ : M/\ker J \otimes J \to MJ/0 = MJ/(\ker J \cap MJ)$$

such that $\phi\zeta = 1_{MJ/(\ker J \cap MJ)}$. Hence $(j \otimes 1)\ \eta\ \phi\ \rho_J \ = \ \rho_{I^{MJ}}$ is an $R-$isomorphism, from which we have an $R-$monomorphism $j \otimes 1$. By combining this with the surjectivity of $j \otimes 1$, $j \otimes 1$ becomes an $R-$isomorphism. And also the homomorphism

$$1_{M/\ker J} \otimes \iota \ : M/\ker J \otimes_S J \to M/\ker J \otimes_S I^{MJ}$$

is an $R-$isomorphism since $1_{M/\ker J} \otimes \iota = (j \otimes 1_J)^{-1}(1 \otimes \iota)(j \otimes 1)$ is the composition of isomorphisms.

It remains to show that $\ker J \subseteq \ker I^{MJ}$. For each $m \in \ker J$, the fact of

$$(m + \ker J) \otimes g = 0_{M/\ker J \otimes I^{MJ}}\ , \ \text{ for every } \ g \in I^{MJ}$$

says that $mg = 0$ always for each $g \in I^{MJ}$. Thus $\ker J \subseteq \ker I^{MJ}$ follows. Therefore the *cosimilarity* of J and I^{MJ} follows. Hence the proof is completed. □

REMARK 2.3 ([6]). For an *endo-flat* module ${}_RM$, we have the following:

(1) For an *open* submodule L and a submodule L', $I^L = I^{L'}$ implies that $L \leq L'$.
(2) For a *closed* submodule N and a submodule N', $I_N = I_{N'}$ implies $N' \leq N$.
(3) But the converse of the above Theorem 2.2 doesn't hold, in general. A left $\mathbb{Z}-$module itself ${}_{\mathbb{Z}}\mathbb{Z}$ tells immediately that the *cosimilarity* doesn't imply the *similarity*, where $\mathbb{Z}$ is the ring of integers.
(4) On an *endo-flat* module M, for each left ideal $J \trianglelefteq_l End_R(M)$, the *closed* submodule ker J is *open* by Lemma 1.2. Hence we have that

$$\{ H \leq M \mid H \text{ is a } \textit{closed} \text{ submodule of } M \} \subseteq \{ K \leq M \mid K \text{ is an } \textit{open} \text{ submodule of } M \}.$$

The items of the following Remark 2.4 proved in [6] are restated for good insights of relationships of chain conditions.

REMARK 2.4. ([6]) Let $[J]$ be the equivalence class containing J in the set $\{ J \trianglelefteq_l End_R(M) \}/_{sim\sim}$.

(1) If an *endo-flat* module M is *self-generated*, then we have one-to-one correspondences between the following sets:

$$\{ J \leq End_R(M) \mid J \trianglelefteq_l End_R(M) \}/_{sim\sim} = \{ [J] \mid J \trianglelefteq_l End_R(M) \} \overset{1-1}{\longleftrightarrow} \{ A \leq M \} \overset{1-1}{\longleftrightarrow} \{ I^A \mid A \leq M \}.$$

(2) If an *endo-flat* module M is *self-cogenerated*, then we have the following:

$$\begin{aligned} \{L \leq M\} &= \{ A \leq M \mid A \text{ is } \textit{open} \} \\ &= \{ B \leq M \mid B \text{ is } \textit{closed} \} \\ &\supseteq \{ B \leq M \mid B \text{ is } \textit{closed fully invariant} \}. \end{aligned}$$

(3) Let (J) be the equivalence class containing J in the set $\{ J \trianglelefteq_l End_R(M) \}/_{cosim\simeq}$.

If an *endo-flat* module M is *self-cogenerated*, then there are one-to-one correspondences between the following sets:

$$\begin{aligned}\{J \trianglelefteq_l End_R(M)\}/_{cosim\simeq} &= \{ (J) \mid J \trianglelefteq_l End_R(M) \} \\ &\overset{1-1}{\longleftrightarrow} \{ B \leq M \mid B \text{ is } \textit{fully invariant} \} \\ &\overset{1-1}{\longleftrightarrow} \{ I_B \mid B \leq M \text{ is } \textit{fully invariant} \}.\end{aligned}$$

3. Chain conditions of a faithful endo-flat module whose endomorphism ring has the zero left annihilator of each non-zero endomorphism

The following corollary is an immediate consequence of Theorem 2.2.

COROLLARY 3.1. *For a faithful endo-flat module $_RM$ with the endomorphism ring $End_R(M)$ having the zero left annihilator of each non-zero endomorphism, there is a one-to-one function from $\{ J \trianglelefteq_l End_R(M) \}/_{cosim\simeq}$ into $\{ J \trianglelefteq_l End_R(M) \}/_{sim\sim}$.*

PROPOSITION 3.2. *For a faithful endo-flat module $_RM$, if the endomorphism ring $End_R(M)$ has the zero left annihilator of each non-zero endomorphism, then the following are obtained easily:*

(1) *For a self-generated module M, if $End_R(M)$ is left Noetherian, then M is Noetherian.*
(2) *For a self-generated module M, if $End_R(M)$ is left Artinian, then M is Artinian.*
(3) *For a self-cogenerated module M, if $End_R(M)$ is left Noetherian, then M is Artinian and Noetherian.*
(4) *For a self-cogenerated module M, if $End_R(M)$ is left Artinian, then M is Artinian and Noetherian.*
(5) *For a self-cogenerated module M, if $End_R(M)$ is right Noetherian, then M is Artinian.*
(6) *For a self-cogenerated module M, if $End_R(M)$ is right Artinian, then M is Noetherian.*

PROOF. For (1), (2), (3) and (4) the proofs are easy so we will not write them here. For (5), let

$$N_1 \geq N_2 \geq \cdots \geq N_n \geq N_{n+1} \geq \cdots$$

be any descending chain of submodules of a *self-cogenerated* module M, then we have an ascending chain of right ideals of $End_R(M)$

$$J_1 = I_{N_1} \subseteq J_2 = I_{N_2} \subseteq \cdots \subseteq J_n = I_{N_n} \subseteq J_{n+1} = I_{N_{n+1}} \subseteq \cdots .$$

Since $End_R(M)$ is a right Noetherian, there is an m such that

$$J_m = I_{N_m} = J_{m+i} = I_{N_{m+i}} \quad \text{for all } \ i = 1, 2, 3, \cdots .$$

Then, since the N_i's are *closed* submodules,

$$\ker J_m = N_m = \ker J_{m+i} = N_{m+i} \qquad \text{for all } i = 1, 2, 3, \cdots$$

follows immediately from $I_{N_m} = I_{N_{m+i}}$ for all $i = 1, 2, 3, \cdots$. Hence M is Artinian. For (6), proof follows by taking the reversing inclusion and the right Artinian ring $End_R(M)$ in the previous item (5).

The theorem stated on page 69 in [2] is well known: If $End_R(M)$ is right Artinian, then any right $End_R(M)$-module is Noetherian if, and only if, it is Artinian. □

Combining the above theorem with the facts;

$$\begin{aligned}\{A \leq M \mid A \text{ is } \textit{open}\} &\supseteq \{ B \leq M \mid B \text{ is } \textit{closed} \} \\ &\supseteq \{ B \leq M \mid B \text{ is } \textit{closed fully invariant} \},\end{aligned}$$

we have the following theorem.

THEOREM 3.3. *If a faithful endo-flat module* $_RM$ *is self-cogenerated and if the endomorphism ring* $End_R(M)$ *has the zero left annihilator of each non-zero endomorphism, then* M *is Artinian if and only if it is Noetherian.*

PROOF. Assume that M is a Noetherian module. Let

$$N_1 \geq N_2 \geq \cdots \geq N_n \geq N_{n+1} \geq \cdots$$

be any descending chain of submodules of M. Then we have an ascending chain of right ideals of $End_R(M)$:

$$J_1 = I_{N_1} \subseteq J_2 = I_{N_2} \subseteq \cdots \subseteq J_n = I_{N_n} \subseteq J_{n+1} = I_{N_{n+1}} \subseteq \cdots,$$

from which we have an ascending chain of submodules of M:

$$MJ_1 \leq MJ_2 \leq \cdots \leq MJ_n \leq MJ_{n+1} \leq \cdots.$$

Since M is Noetherian, there is an m such that

$$MJ_m = MJ_{m+i} \qquad \text{for all } i = 1, 2, 3, \cdots .$$

Thus J_m and J_{m+i} are *similar*, so J_m and J_{m+i} are *cosimilar* for all $i = 1, 2, 3, \cdots$, by Theorem 2.5. In other words,

$$\ker J_m = N_m = \ker J_{m+i} = N_{m+i} \cdots \quad \text{for all } i = 1, 2, 3, \cdots .$$

Hence M is Artinian. Conversely, assume that M is an Artinian module. Let

$$N_1 \leq N_2 \leq \cdots \leq N_n \leq N_{n+1} \leq \cdots$$

be an ascending chain of submodules of M. We have a descending chain of right ideals of $End_R(M)$:

$$J_1 = I_{N_1} \supseteq J_2 = I_{N_2} \supseteq \cdots \supseteq J_n = I_{N_n} \supseteq J_{n+1} = I_{N_{n+1}} \supseteq \cdots,$$

from which we have a descending chain of submodules of M;

$$ImJ_1 \geq ImJ_2 \geq \cdots \geq ImJ_n \geq ImJ_{n+1} \geq \cdots .$$

Since M is Artinian, there is an m such that

$$ImJ_m = ImI_{N_m} = ImJ_{m+i} = ImI_{N_{m+i}} \qquad \text{for all } i = 1, 2, 3, \cdots .$$

Thus J_m and J_{m+i} are *similar* for every $i = 1, 2, 3, \cdots$. Hence by Theorem 2.2, J_m and J_{m+i} are *cosimilar*. Since M is *endo-flat* and *self-cogenerated*, every submodule of M is *closed* and *open*. Thus

$$\ker J_m = N_m = \ker J_{m+i} = N_{m+i} \qquad \text{for all } i = 1, 2, 3, \cdots,$$

which implies that M is Noetherian. Hence the proof is completed. □

The following corollary is an immediate consequence of the Proposition 3.2 and Theorem 3.3.

COROLLARY 3.4. *If a left R–faithful module ${}_RM$ is self-cogenerated endo-flat and if its endomorphism ring $End_R(M)$ has the zero left annihilator of each non-zero endomorphism, then we have the following:*

(1) *If $End_R(M)$ is a left (or right, or two-sided) Noetherian ring, then M is Artinian and Noetherian.*
(2) *If $End_R(M)$ is a left (or right, or two-sided) Artinian ring, then M is Artinian and Noetherian.*

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References

[1] M. F. Atyah Frs. and I. G. Macdonald, *Introduction of Commutative Algebra*, Addison-Wesley Publ. Comp., 1969.
[2] J. Lambek, *Lectures on Rings and Modules* (2nd, ed.), Chelsa Publ. Comp. New York, NY, 1976.
[3] V. P. Camillo and K. R. Fuller, *Rings whose faithful modules are flat over their endomorphism rings*, vol. 27, Arch. Math., 1976, pp. 522-525.
[4] S. M. Khuri, *Correspondence theorems for modules and their endomorphism rings*, Journal of Algebra **122** (1989), 380-396.
[5] F. W. Anderson and K. R. Fuller, *Rings and categories of modules* (2nd, ed.), Springer-Verlag, 1992.
[6] Soon-Sook Bae, *On the Chain conditions of the endomorphism ring and of a flat module*, Pusan Kyŏngnam Mathematical Journal **12** (1996), no. 2, 213-233.

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