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FIXED POINT THEOREMS FOR λ-FIRMLY NONEXPANSIVE MAPPINGS IN METRIC SPACES OF HYPERBOLIC TYPE

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ABSTRACT. In this paper, we prove that any λ-firmly nonexpansive mapping $(0 < \lambda < 1)$ $T : C \to C$ has a fixed point in C whenever C is a finite union of nonempty, bounded, closed and convex subsets of a metric space of hyperbolic type.

1. Introduction

We suppose that (M, d) is a metric space containing a family L of metric lines such that distinct points $x, y \in M$ lie on exactly one number $l(x,y)$ of L. This metric line determines a unique metric segment joining x and y. We denote this segment by $S[x, y]$. For each $\alpha \in [0,1]$ there is a unique point z in $S[x,y]$ for which

$$d(x,z) = \alpha d(x,y) \text{ and } d(z,y) = (1-\alpha)d(x,y)$$

Adopting the notation of [8] or [17], we shall denote this point by $(1-\alpha)x \oplus \alpha y$.

We shall say that M, or more precisely (M, d, L), is a *hyperbolic space* if

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}d(y,z)$$

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for all x, y and z in M.

In section 2 of this paper, we propose the various classes of metric spaces, especially, metric spaces of pre-hyperbolic type. We shall find the equivalent convexity condition for a complete metric space of pre-hyperbolic type (see Proposition 2.1). In section 3, we prove that any λ-firmly nonexpansive mapping $(0 < \lambda < 1)$ $T : C \to C$ has a fixed point in C whenever C is a finite union of nonempty, bounded, closed and convex subsets of a complete metric space of pre-hyperbolic type (see Theorem 3.2).

2. Spaces of hyperbolic type

DEFINITION 2.1. A metric space M is said to be of *pre-hyperbolic* type if for each $x, y \in M$ there is a specified metric segment $S[x, y]$ joining x and y, which has the property that if $p \in M$ and if m is the point of $S[x, y]$ which satisfies $d(x, m) = \alpha d(x, y)$, then

$$d(p, m) \leq (1 - \alpha) d(p, x) + \alpha d(p, y). \tag{A}$$

DEFINITION 2.2. A metric space M is said to be of hyperbolic type if for each $x, y \in M$ there is a specified metric segment $S[x, y]$ joining x and y for which the following property holds: Let $p, q, r \in M$ and $\alpha \in (0, 1)$, and suppose m_1 and m_2 are points of $S[p, r]$ and $S[p, q]$ respectively, which satisfy

$$d(m_1, p) = \alpha d(p, r) \text{ and } d(m_2, p) = \alpha d(p, q).$$

Then

$$d(m_1, m_2) \leq \alpha d(r, q). \tag{H}$$

Obviously, (H) implies (A) (cf. [13]). There is an important consequence of condition (H). If M is of hyperbolic type and if $m_1 = (1 - \alpha)p \oplus \alpha q$ and $m_2 = (1 - \alpha)s \oplus \alpha r$, for $p, q, r, s \in M$ and $\alpha \in (0, 1)$, then (H) in fact implies

$$d(m_1, m_2) \leq (1 - \alpha) d(p, s) + \alpha d(q, r). \tag{H'}$$

The following lemma was mentioned in [13].

LEMMA 2.1. *Let (M, d) be a metric space of hyperbolic type. Then (H) is equivalent to the following property*

$$(\mathrm{H_0}) \qquad d(\frac{1}{2}p \oplus \frac{1}{2}r, \frac{1}{2}p \oplus \frac{1}{2}q) \leq \frac{1}{2}d(r, q)$$

for all p, r and q in M.

We remark that the term 'hyperbolic type' is used in the above context because condition (H) with strict inequality is characteristic of hyperbolic geometry (see [23]). At the same time, all normed linear spaces are of hyperbolic type. (As a matter of fact, if equality always holds in (H), then the resulting condition *characterizes* normed linear spaces among an appropriate class of metric spaces ([1])). So are all Hadamard manifolds, that is, all finite-dimensional connected, simply connected, complete Riemannian manifolds of nonpositive curvature (cf., [4, pp. 305]). An infinite-dimensional example is provided by the Hilbert ball equipped with the hyperbolic metric (see [8, pp, 104]). For other results in this setting we refer, for example, to Reich [18] (and citations therein), Shafrir [20] and Reich and Shafir [17].

DEFINITION 2.3. A metric space M is *strongly convex* provided that for any two points $x, y \in M$ there is only one point $z \in M$ such that

$$d(x, z) = d(y, z) = \frac{1}{2}d(x, y).$$

Such a z will be called a *strong midpoint* of x and y.

Obviously, every strongly convex complete metric space (M, d) yields a unique metric segment $S[x, y]$ for each $x, y \in M$. Therefore, every hyperbolic metric space is strongly convex. For a characterization of the metric space of pre-hyperbolic type, we need the following stronger concept.

DEFINITION 2.4. A metric space (M, d) is said to have strongly convex ball intersections if for each $x, y \in M$

$$\bigcap_{u \in M} B(u; \frac{1}{2}d(x, u) + \frac{1}{2}d(y, u)) \neq \emptyset.$$

It is easy to see that all $z \in \bigcap_{u \in M} B(u, \frac{1}{2}d(x,u) + \frac{1}{2}d(y,u))$ are strong midpoints of x and y. Also, every metric space of pre-hyperbolic type has strongly convex ball intersections. It is natural to ask whether the converse holds or not. The following gives an affirmative answer if M is complete.

PROPOSITION 2.1. *Let (M,d) be a complete metric space. Then M has strongly convex ball intersections if and only if it is of pre-hyperbolic type.*

PROOF. It suffices to show "only if". Let $x_0, x_1 \in M$, $x_0 \neq x_1$. Since M has strongly convex ball intersections, x_0 and x_1 have a strong midpoint $x_{\frac{1}{2}}$, i.e.,

$$x_{\frac{1}{2}} \in \bigcap_{u \in M} B(u, \frac{1}{2}d(x_0, u) + \frac{1}{2}d(x_1, u)).$$

Similarly, there exists points $x_{1/4}, x_{3/4}$ which are respective strong midpoints of $(x_0, x_{\frac{1}{2}})$ and $(x_{\frac{1}{2}}, x_1)$. Note that if $p \in M$ and if $i = 1, 3$ then

$$d(p, x_{i/4}) \leq (1 - \frac{i}{4})d(p, x_0) + \frac{i}{4}d(p, x_1).$$

The idea is proceed by induction. Letting $\rho = d(x_0, x_1)$ and mimicking the proof of [7; pp.25–26] (only replaced d by ρ in [7]), we know that the closure of the set

$$\bigcup_{n=1}^{\infty} \{x_{k/2^n} : 1 \leq k \leq 2^n - 1\}$$

is the desired metric segment $S[x_0, x_1]$ joining x_0 and x_1, which clearly satisfies (A).

3. Fixed point theorems

Let (M,d) be a metric space of hyperbolic type. The *modulus of convexity* $\delta : (0,\infty) \times (0,2] \to [0,1]$ of M is defined by setting

$$\delta(r,\epsilon) = \inf\Big\{1 - \frac{1}{r}d(a, \frac{1}{2}x \oplus \frac{1}{2}y)\Big\}$$

where the infimum is taken over all points a, x and y satisfying $d(a,x) \leq r$, $d(a,y) \leq r$ and $d(x,y) \geq r\epsilon$. We say that M is *uniformly convex* if δ is always positive. Several examples of uniformly convex metric spaces of hyperbolic type are given in [8] and [17]. In particular the infinite dimensional complex Hilbert unit ball is a uniformly convex metric spaces of hyperbolic type. In this space, we know [8, pp.107] that

$$\delta(r,\epsilon) = 1 - \frac{1}{r}\tan^{-1}\frac{[\sinh(r(1+\epsilon/2))\sinh(r(1-\epsilon/2))]^{1/2}}{\cosh r}$$

Therefore, $\delta(r,\epsilon)$ is continuous on $(0,\infty) \times (0,2]$. For our argument of the general metric space of pre-hyperbolic type, throughout this section, we assume that $\delta(r,\epsilon)$ is also continuous on $(0,\infty) \times (0,2]$. The fixed point theory on metric spaces of hyperbolic type have been studied widely (see [8],[12],[13]). We will say that a subset C of a metric space M of pre-hyperbolic type is *convex* if $S[x,y] \subset C$ whenever $x, y \in C$.

Let $\{x_n\}$ be a bounded sequence in a metric space (M,d), and let C be a closed convex subset of M. Consider the functional $f : C \to [0,\infty)$ defined by

$$f(x) = \limsup_{n\to\infty} d(x_n, x) \tag{3.1}$$

for all $x \in M$.

The infimum of $f(x)$ over C is called the *asymptotic radius* of $\{x_n\}$ with respect to C. A point z in C is called the *asymptotic center* of $\{x_n\}$ with respect to C if

$$f(z) = \inf\{f(x) : x \in C\}$$

The set of all asymptotic center is denoted by $A(C, \{x_n\})$

We can obtain the following lemma with a similar manner of uniformly convex Banach space (see Theorem 4.1 and Proposition 18.1 of [8]).

LEMMA 3.1. *Let (M,d) be a uniformly convex complete metric space M of pre-hyperbolic type. Then every bounded sequence in M has a unique asymptotic center with respect to any closed convex subset of M, i.e. $A(C,\{x_n\}) = \{z\}$.*

The following useful result is a direct consequence of Lemma 3.1. This is a natural hyperbolic metric version of Corollary 1 of [16] in a uniformly convex Banach space.

LEMMA 3.2. *Let (M,d) be a uniformly convex complete metric space M of pre-hyperbolic type. Let $\{x_n\}$ be a bounded sequence in a closed convex subset C of M and $A(C,\{x_n\}) = \{z\}$. Then*

$$\{y_m\} \subset C \text{ and } \lim_{m\to\infty} f(y_m) = f(C,\{x_n\}) \Rightarrow \lim_{m\to\infty} y_m = z,$$

where $f(C,\{x_n\})$ means the asymptotic radius of $\{x_n\}$ with respect to C.

Let (M,d) be a metric space and $C \subset M$. Let $T : C \to C$ be a self-mapping of C. There appear in the literature two definitions of an asymptotically nonexpansive mapping. The weaker definition (cf., Kirk [15]) requires that for each $x \in C$ $\lim_{n\to\infty} c_n(x) = 0$, where

$$c_n(x) := \max\{0, \sup_{y\in C}[d(T^n x, T^n y) - d(x,y)]\}.$$

Such a mapping is later said to be of asymptotically nonexpansive type. The stronger definition (briefly called *asymptotically nonexpansive* as in [5]) requires each iterate T^n to be Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. Every nonexpansive mapping is asymptotically nonexpansive. All asymptotically nonexpansive mappings are Lipschitzian, but mappings of asymptotically nonexpansive type is not Lipschitzian.

We obtain the following theorems with the similar manner. The following results are well-known facts in uniformly convex Banach spaces. Compare Theorem 5.2 of [8] for nonexpansive mappings, and Theorem 1 of [10] for asymptotically nonexpansive mappings.

THEOREM 3.1. *Let C be a closed and convex subset of a uniformly convex complete metric space (M, d) of pre-hyperbolic type. If $T : C \to C$ is a continuous mapping of asymptotically nonexpansive type. Then T has a fixed point if and only if there exists a point $x \in C$ such that the sequence of iterates $\{T^n x\}$ is bounded.*

PROOF. The proof is mimicking the lines of the proof of [10] by using Lemma 3.1 and 3.2. Since the necessity follows easily, it suffices to show "if". Assume $x_0 \in C$ is such that the sequence $\{x_n = T^n x_0\}$ is bounded, and let $A(C, \{x_n\}) = \{z\}$. Let $\{y_m = T^m z\}$. We shall show

$$f(y_m) = \limsup_{n\to\infty} d(x_n, y_m) \to f(C, \{x_n\}) := r \text{ as } m \to \infty.$$

By lemma 3.2, this would imply $y_m \to z$ as $m \to \infty$, and because T is continuous

$$Tz = T(\lim_{m\to\infty} T^m z) = \lim_{m\to\infty} T^{m+1} z = z.$$

For two integers $n > m \geq 1$ we have

$$d(x_n, y_m) = d(T^m x_{n-m}, T^m z) \leq c_m(z) + d(x_{n-m}, z).$$

Taking lim sup as $n \to \infty$ on both sides, this implies $r \leq f(y_m) \leq c_m(z) + f(z) = c_m(z) + r$ and so $\lim_{m\to\infty} f(y_m) = r$.

The following is a natural partial metric version of [10,Theorem 1] in a Banach space.

COROLLARY 3.1 *Let C be a closed and convex subset of a uniformly convex complete metric space (M, d) of pre-hyperbolic type. If $T : C \to C$ is a asymptotically nonexpansive mapping Then the following are equivalent*

(a) *T has a fixed point.*
(b) *There exists a point $x \in C$ such that the sequence of iterates $\{T^n x\}$ is bounded.*
(c) *There exists a bounded approximating sequence $\{x_n\}$ for T.*

DEFINITION 3.1. Let C be a nonempty subset of a metric space (M, d) of hyperbolic type, and let $\lambda \in (0,1)$. Then $T : C \to M$ is said to be *λ-firmly nonexpansive* if

$$d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty)$$

for all $x, y \in C$.

See [8] in a Banach space case. It is easily seen that λ-firmly nonexpansive is nonexpansive. Conversely, to each nonexpansive $T : C \to C$ one can associate a firmly nonexpansive mapping with the same fixed point set whenever C is closed and convex (cf., see [8,pp. 124] in the Hilbert ball with the hyperbolic metric) Moreover, from the point of view of fixed point theory for the class of all closed convex subsets C, firmly nonexpansive mappings $T : C \to C$ do not exhibit better behavior than nonexpansive mappings in general [7]. However, this behavior is completely different in the class of nonconnected subsets C in a Banach space setting (cf.,[21]).

For our further argument, we *suppose* M satisfies the following property:

(S) If $d(a, x) = d(a, y) := \gamma > 0$ and if $a \in S[u_\lambda, v_\lambda]$ for some $\lambda \in (0,1)$, then either $a \in S[x, y]$ or $x = y$, where $u_\lambda := (1-\lambda)a \oplus \lambda x$ and $v_\lambda := \lambda a \oplus (1-\lambda)y$.

It is easy to see that if X is a strictly convex Banach space, the above property (S) is easily satisfied.

THEOREM 3.2. *Let (M, d) be a uniformly convex complete metric space of hyperbolic type with the property* (S). *Let $C = \cup_{k=1}^{n} C_k$ be a union of nonempty bounded, closed convex subsets C_k of M. Suppose $T : C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0,1)$. Then T has a fixed point in C.*

PROOF. The idea follows the proof of [21]. Let $z \in C$, and let x_k be the asymptotic center of sequence $\{T^i z\}$ with respect to the bounded, closed, convex subsets C_k $(1 \leq k \leq n)$. From Lemma 3.1, the point x_k

is uniquely determined by the identity

$$f(x_k) = \inf_{x \in C_k} f(x), \tag{3.2}$$

where the functional $f : M \to [0, \infty)$ defined by $f(x) = \limsup_{i \to \infty} d(T^i z, x)$. Since T is nonexpansive, we have $d(Tx_k, T^{i+1}z) \leq d(x_k, T^i z)$. Hence

$$f(Tx_k) \leq f(x_k), \tag{3.3}$$

for all k. Now, if $Tx_k \in C_k$ for some k, then the uniqueness of asymptotic center of x_k in conjunction with (3.2) and (3.3) yields $Tx_k = x_k$, which completes the proof. Otherwise, $Tx_k \notin C_k$ for all k, then there exist integers $\{n_1, n_2, \cdots, n_m\} \subset \{1, 2, \cdots, n\}$ $(m \geq 2)$ such that $Tx_{n_k} \in C_{k+1}$ $(k = 1, 2, \cdots m-1)$ and $Tx_{n_m} \in C_1$ Clearly, without loss of generality, one can rearrange the sequence C_k in such a way that $n_k = k$ for all k. Then we have $Tx_k \in C_{k+1}$ $(k = 1, 2, \cdots m-1)$ and $Tx_m \in C_1$ and $x_k \in C_k$ for all k. Hence , one can combine (3.2) and (3.3) in order to get

$$\begin{aligned} f(x_1) &\leq f(Tx_m) \leq f(x_m) \leq f(Tx_{m-1}) \\ &\leq f(x_{m-1}) \leq \cdots \leq f(x_2) \leq f(Tx_1) \leq f(x_1). \end{aligned}$$

Thus we have $f(Tx_k) = f(x_{k+1})$ which, in view of the uniqueness of asymptotic center x_k yield

$$x_{k+1} = Tx_k \ \ (k = 1, 2, \cdots, m), \tag{3.4}$$

where we have denoted $x_{m+1} = x_1$ for the latter simplicity. Hence we readily derive

$$\begin{aligned} d(x_1, x_m) &= d(x_{m+1}, x_m) = d(Tx_m, Tx_{m-1}) \leq d(x_m, x_{m-1}) \\ &= d(Tx_{m-1}, Tx_{m-2}) \leq d(x_{m-1}, x_{m-2}) = \cdots \leq d(x_2, x_1) \\ &= d(Tx_1, Tx_m) \leq d(x_1, x_m), \end{aligned}$$

and

$$d(x_1, x_2) = d(x_2, x_3) = \cdots = d(x_{m-1}, x_m) = d(x_m, x_1) := \gamma. \tag{3 5}$$

Clearly, if $\gamma = 0$, then $x_1 = x_2 = Tx_1$ by (3.4). Hence the proof is complete. On the other hand, since T is λ-firmly nonexpansive it follows from (3.4) that

$$\begin{aligned}\gamma = d(x_{k+1}, x_k) &= d(Tx_k, Tx_{k-1}) \\ &\leq d((1-\lambda)x_k \oplus \lambda Tx_k, (1-\lambda)x_{k-1} \oplus \lambda Tx_{k-1}) \\ &\leq d((1-\lambda)x_k \oplus \lambda x_{k+1}, (1-\lambda)x_{k-1} \oplus \lambda x_k) \\ &\leq (1-\lambda)d(x_k, x_{k-1}) + \lambda d(x_{k+1}, x_k) = \gamma\end{aligned}$$

for $k = 2, 3, \cdots, m$.

Let $u_\lambda := (1-\lambda)x_k \oplus \lambda x_{k+1}$ and $v_\lambda := (1-\lambda)x_{k-1} \oplus \lambda x_k$. Since $d(x_k, u_\lambda) = \lambda d(u_\lambda, v_\lambda)$ and $d(x_k, v_\lambda) = (1-\lambda)d(u_\lambda, v_\lambda)$, $x_k \in S[u_\lambda, v_\lambda]$. By property (S), either $x_k \in S[x_{k-1}, x_{k+1}]$ or $x_{k-1} = x_{k+1}$.

Case 1. $x_k \in S[x_{k-1}, x_{k+1}]$. In this case, since $d(x_k, x_{k-1}) = d(x_k, x_{k+1})$, x_k is a midpoint of $S[x_{k-1}, x_{k+1}]$, i.e., $x_k = \frac{1}{2}x_{k-1} \oplus \frac{1}{2}x_{k+1}$ $(2 \leq k \leq m,\ x_{m+1} = x_1)$. Note that $x_k \in S[x_1, x_m]$ for $k = 2, 3, \cdots, m-1$ and (3.5) again yields $\gamma = d(x_1, x_m) = d(x_1, x_2) + \cdots + d(x_{m-1}, x_m) = (m-1)\gamma$. Since $m \geq 2$, this yields $\gamma = 0$ and also $x_1 = x_2 = Tx_1$ by (3.4).

Case 2. $x_{k-1} = x_{k+1}$. We claim that $d(x_k, (1-\lambda)x_k \oplus \lambda x_{k-1}) = 0$. Then $x_k = (1-\lambda)x_k \oplus \lambda x_{k-1}$ and so this with (3.5) gives $x_k = x_{k-1} = x_{k+1} = Tx_k$. Hence T is a fixed point x_k. Suppose that $d(x_k, (1-\lambda)x_k \oplus \lambda x_{k-1}) > 0$. Since

$$\begin{aligned}d(x_{k-1}, x_k) = d(x_{k-1}, (1-\lambda)x_{k-1} \oplus \lambda x_k) \\ + d((1-\lambda)x_{k-1} \oplus \lambda x_k, (1-\lambda)x_k \oplus \lambda x_{k-1}) \\ + d((1-\lambda)x_k \oplus \lambda x_{k-1}, x_k),\end{aligned}$$

we have

$$d((1-\lambda)x_{k-1} \oplus \lambda x_k, (1-\lambda)x_k \oplus \lambda x_{k-1}) < d(x_{k-1}, x_k).$$

Since T is λ-firmly nonexpansive, this implies

$$\begin{aligned}d(x_k, x_{k-1}) = d(x_k, x_{k+1}) &= d(Tx_{k-1}, Tx_k) \\ &\leq d((1-\lambda)x_{k-1} \oplus \lambda Tx_{k-1}, (1-\lambda)x_k \oplus \lambda Tx_k) \\ &\leq d((1-\lambda)x_{k-1} \oplus \lambda x_k, (1-\lambda)x_k \oplus \lambda x_{k-1}) < d(x_k, x_{k-1}).\end{aligned}$$

This is a contradiction. This contradiction proves our claim.

As a direct consequence of Theorem 3.2, we have the following.

COROLLARY 3.2 [21]. *Let X be a uniformly convex Banach space, let $C = \cup_{n=1}^{n} C_k$ be a union of nonempty bounded, closed convex subsets C_k of X. Suppose $T : C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0,1)$. Then T has a fixed point in C.*

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