East Asian Math J 15(1999), No. 2, pp 301-312

EXISTENCE OF ERGODIC RETRACTIONS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS WITH NONCONVEX AND NONCLOSED DOMAINS

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ABSTRACT. In this paper, we study the nonlinear ergodic retraction theorems for an asymptotically nonexpansive semigroup with nonconvex and nonclosed domains in Hilbert spaces

1. Preliminaries and notations

Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let G be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \mapsto s \cdot t$ and $s \mapsto t \cdot s$ of G into itself are continuous. Let C be a nonempty subset of H and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup on C, i.e., $S(st)x = S(s)S(t)x$ for all $s, t \in G$ and $x \in C$. Recall that a semigroup $\mathcal{S}$ is said to be an *asymptotically nonexpansive semigroup on C* if each $t \in G$, there exists $k_t > 0$ such that

$$\|S(t)x - S(t)y\| \leq (1 + k_t)\|x - y\|$$

for all $x, y \in C$, where $\inf_{s\in G} \sup_{t\in G} k_{ts} = 0$. In particular, if $k_t = 0$ for all $t \in G$, then $\mathcal{S}$ is called a *nonexpansive semigroup on C*

Received June 7, 1999

1991 Mathematics Subject Classification 47H09, 47H10, 47H20

Key words and phrases Nonlinear ergodic theorem, general semitopological semigroup, asymptotically nonexpansive semigroup, ergodic retraction.

The first author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1998 and the second author was supported by the Research Foundation of Kyungnam University in 1998.

In 1975, Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space : Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. If the set $\mathcal{F}(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro mean

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to a point of $\mathcal{F}(T)$ as $n \to \infty$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $\mathcal{F}(T)$ such that $PT = TP = P$ and $Px \in \overline{conv}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{conv}A$ is the closure of the convex hull of A. The analogous results are given for nonexpansive semigroups on C by Baillon [2] and Brézis-Browder [3]. In [7], Mizoguchi-Takahashi proved a nonlinear ergodic retraction theorem for lipschitzian semigroups by using the notion of submean. And also, in 1992, Takahashi [12] proved the ergodic theorem for nonexpansive semigroups on condition that $\bigcap_{s\in G} \overline{conv}\{S(st)x : t \in G\} \subset C$ for some $x \in C$.

In this paper, without using the concept of submean, we attempt to prove the nonlinear ergodic retraction theorems for an asymptotically nonexpansive semigroup with nonconvex and nonclosed domains in a Hilbert space. For the proof of main theorem, we prove that if C is a nonempty subset of a Hilbert space H, G a semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ a representation of G as asymptotically nonexpansive mappings of C into itself, then

$$\bigcap_{s\in G} \overline{conv}\{S(ts)x : t \in G\} \bigcap E(\mathcal{S})$$

is at most one point, where

$$E(x) = \left\{ z : \inf_{s\in G} \sup_{t\in G} \|S(ts)x - z\| = \inf_{t\in G} \|S(t)x - z\| \right\}$$

and

$$E(\mathcal{S}) = \bigcap_{x\in C} E(x).$$

Our results are generalizations and improvements of the known results of Baillon [1], Brézis-Browder [3], Hirano-Takahashi [4], Ishihara-Takahashi [5], Lau-Nishiure-Takahashi [6], Mizoguchi-Takahashi [7], Reich [8], Takahashi-Zhang [9], and Takahashi ([10], [11] and [12]) in many directions.

2. Lemma and proposition

Throughout this paper, we assume that C is a nonempty subset of a real Hilbert space H, G a semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. For each $x \in C$, define $E(x)$ and $E(\mathcal{S})$ by

$$E(x) = \left\{ z : \inf_{s \in G} \sup_{t \in G} \|S(ts)x - z\| = \inf_{t \in G} \|S(t)x - z\| \right\}$$

and

$$E(\mathcal{S}) = \bigcap_{x \in C} E(x)$$

respectively. And we denote $\mathcal{F}(\mathcal{S})$ by the set $\{x \in C : S(s)x = x \text{ for all } s \in G\}$ of common fixed points of $\mathcal{S}$.

We begin with the following important lemma and proposition for our main theorems.

LEMMA 2.1. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. Then we have $\mathcal{F}(\mathcal{S}) \subset E(\mathcal{S})$.*

PROOF. Let $x \in C$ and $f \in \mathcal{F}(\mathcal{S})$. Since $\mathcal{S}$ is an asymptotically nonexpansive semigroup, for an $\varepsilon > 0$, there exists $s_0 \in G$ such that for all $t \in G$

$$k_{ts_0} < \varepsilon.$$

Hence, for each $a \in G$,

$$\begin{aligned} \inf_{s \in G} \sup_{t \in G} \|S(ts)x - f\| &\leq \sup_{t \in G} \|S(ts_0 a)x - f\| \\ &\leq \sup_{t \in G} (1 + k_{ts_0}) \|S(a)x - f\| \\ &\leq (1 + \varepsilon)\|S(a)x - f\| \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\inf_{s\in G}\sup_{t\in G}\|S(ts)x - f\| \leq \inf_{t\in G}\|S(t)x - f\|.$$

Therefore $f \in E(x)$ for all $x \in C$. This completes the proof.

The following proposition plays a crucial role in the proof of our main theorem in this paper.

PROPOSITION 2.2. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. Then for every $x \in C$, the set*

$$\bigcap_{s\in G}\overline{conv}\{S(ts)x : t \in G\}\bigcap E(x)$$

consists of at most one point.

PROOF. Let $u, v \in \bigcap_{s\in G}\overline{conv}\{S(ts)x : t \in G\}\bigcap E(x)$. Without loss of generality, we may assume that

$$\inf_{t\in G}\|S(t)x - u\|^2 \leq \inf_{t\in G}\|S(t)x - v\|^2.$$

Now, for each $t, s \in G$, since

$$\|u - v\|^2 + 2(S(ts)x - u, u - v) = \|S(ts)x - v\|^2 - \|S(ts)x - u\|^2,$$

we have

$$\begin{aligned}\|u - v\|^2 &+ 2\inf_{t\in G}(S(ts)x - u, u - v)\\ &\geq \inf_{t\in G}\|S(ts)x - v\|^2 - \sup_{t\in G}\|S(ts)x - u\|^2\\ &\geq \inf_{t\in G}\|S(t)x - v\|^2 - \sup_{t\in G}\|S(ts)x - u\|^2.\end{aligned}$$

Since $u \in E(x)$,

$$\begin{aligned}\|u-v\|^2 &+ 2\sup_{s\in G}\inf_{t\in G}(S(ts)x-u,u-v)\\ &\geq \inf_{t\in G}\|S(t)x-v\|^2 - \inf_{s\in G}\sup_{t\in G}\|S(ts)x-u\|^2\\ &= \inf_{t\in G}\|S(t)x-v\|^2 - \inf_{t\in G}\|S(t)x-u\|^2\\ &\geq 0.\end{aligned}$$

Let $\varepsilon > 0$. Then there is an $s_1 \in G$ such that

$$\|u-v\|^2 + 2(S(ts_1)x-u,u-v) > -\varepsilon$$

for all $t \in G$. From $v \in \overline{conv}\{S(ts_1)x : t \in G\}$, we have

$$\|u-v\|^2 + 2(v-u,u-v) \geq -\varepsilon.$$

This implies that $\|u-v\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $u = v$. This completes the proof.

3. Ergodic retraction theorems

Now, we are in a position to establish the ergodic retraction theorems for asymptotically nonexpansive semigroups with nonconvex and nonclosed domains in a Hilbert space.

THEOREM 3.1. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C such that $E(\mathcal{S}) \neq \emptyset$. Then the following statements are equivalent:*

(1) $\bigcap_{s\in G}\overline{conv}\{S(ts)x : t \in G\}\bigcap E(\mathcal{S}) \neq \emptyset$ *for each* $x \in C$.
(2) *There is a unique nonexpansive retraction P of C into $E(\mathcal{S})$ such that $PS(t) = P$ for every $t \in G$ and $Px \in \overline{conv}\{S(t)x : t \in G\}$ for every $x \in C$.*

PROOF. (2) $\Rightarrow$ (1). We know that $Px \in E(\mathcal{S})$ for each $x \in C$. And also $Px \in \bigcap_{s\in G} \overline{conv}\{S(ts)x : t \in G\}$. In fact, for all $s \in G$,

$$\begin{aligned} Px = PS(s)x &\in \overline{conv}\{S(t)S(s)x : t \in G\} \\ &= \overline{conv}\{S(ts)x : t \in G\}. \end{aligned}$$

(1) $\Rightarrow$ (2). By Proposition 2.2, $\bigcap_{s\in G} \overline{conv}\{S(ts)x : t \in G\} \bigcap E(\mathcal{S})$ contains exactly one point for each $x \in C$, say it Px. Hence, for each $a \in G$,

$$\begin{aligned} \{PS(a)x\} &= \bigcap_{s\in G} \overline{conv}\{S(tsa)x : t \in G\} \bigcap E(\mathcal{S}) \\ &\supseteq \bigcap_{s\in G} \overline{conv}\{S(ts)x : t \in G\} \bigcap E(\mathcal{S}) \\ &= \{Px\}. \end{aligned}$$

This implies that $PS(t) = P$ for every $t \in G$.

Finally, we have to show that P is nonexpansive. Let $x, y \in C$ and $0 < \lambda < 1$. Since $Py \in E(\mathcal{S})$, for any $\varepsilon > 0$, there exists $s_1 \in G$ such that

$$\sup_{t\in G} \|S(ts_1)x - Py\| \leq \inf_{t\in G} \|S(t)x - Py\| + \varepsilon.$$

Therefore, we have

$$\begin{aligned} &\|\lambda S(tss_1)x + (1-\lambda)Px - Py\|^2 \\ &\qquad = \|\lambda(S(tss_1)x - Py) + (1-\lambda)(Px - Py)\|^2 \\ &\qquad = \lambda\|S(tss_1)x - Py\|^2 + (1-\lambda)\|Px - Py\|^2 \\ &\qquad\quad - \lambda(1-\lambda)\|S(tss_1)x - Px\|^2 \\ &\qquad \leq \lambda(\|S(ab)x - Py\| + \varepsilon)^2 + (1-\lambda)\|Px - Py\|^2 \\ &\qquad\quad - \lambda(1-\lambda) \inf_{t\in G} \|S(t)x - Px\|^2, \end{aligned}$$

for each $t, s, a, b \in G$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\|\lambda S(ts)x + (1-\lambda)Px - Py\|^2 \\
&\quad\le \lambda\|S(ab)x - Py\|^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|S(t)x - Px\|^2 \\
&\quad= \|\lambda S(ab)x + (1-\lambda)Px - Py\|^2 + \lambda(1-\lambda)\|S(ab)x - Px\|^2 \\
&\qquad - \lambda(1-\lambda)\inf_{t\in G}\|S(t)x - Px\|^2.
\end{aligned}$$

It is then easily seen that

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\|\lambda S(ts)x + (1-\lambda)Px - Py\|^2 - \lambda(1-\lambda)\inf_{b\in G}\sup_{a\in G}\|S(ab)x - Px\|^2 \\
&\le \sup_{b\in G}\inf_{a\in G}\|\lambda S(ab)x + (1-\lambda)Px - Py\|^2 - \lambda(1-\lambda)\inf_{t\in G}\|S(t)x - Px\|^2.
\end{aligned}$$

Since $Px \in E(\mathcal{S})$, we have

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\|\lambda S(ts)x + (1-\lambda)Px - Py\|^2 \\
&\qquad\le \sup_{s\in G}\inf_{t\in G}\|\lambda S(ts)x + (1-\lambda)Px - Py\|^2.
\end{aligned} \tag{1}$$

Let

$$h(\lambda) = \inf_{s\in G}\sup_{t\in G}\|\lambda S(ts)x + (1-\lambda)Px - Py\|^2$$

Then for any $\varepsilon > 0$, there exists $s_2 \in G$ such that for all $t \in G$,

$$\|\lambda S(ts_2)x + (1-\lambda)Px - Py\|^2 \le h(\lambda) + \varepsilon$$

and hence

$$(\lambda S(ts_2)x + (1-\lambda)Px - Py, Px - Py) \le (h(\lambda) + \varepsilon)^{\frac{1}{2}}\|Px - Py\|$$

for all $t \in G$. Since $Px \in \overline{conv}\{S(ts_2)x : t \in G\}$, we have

$$(\lambda Px + (1-\lambda)Px - Py, Px - Py) \le (h(\lambda) + \varepsilon)^{\frac{1}{2}}\|Px - Py\|.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\|Px - Py\|^2 \leq h(\lambda),$$

that is

$$\|Px - Py\|^2 \leq \inf_{s\in G}\sup_{t\in G}\|\lambda S(ts)x + (1-\lambda)Px - Py\|^2. \tag{2}$$

Now, one can choose an $s_3 \in G$ such that $\|S(ts_3)x - Px\| \leq M$ for all $t \in G$, where $M = 1 + \inf_{t\in G}\|S(t)x - Px\|$. Then, we have

$$\begin{aligned}\|\lambda S(tss_3)x + (1-\lambda)Px - Py\|^2 &= \|\lambda(S(tss_3)x - Px) + (Px - Py)\|^2 \\ &= \lambda^2\|S(tss_3)x - Px\|^2 + \|Px - Py\|^2 \\ &\quad + 2\lambda(S(tss_3)x - Px, Px - Py) \\ &\leq M^2\lambda^2 + \|Px - Py\|^2 \\ &\quad + 2\lambda(S(tss_3)x - Px, Px - Py).\end{aligned}$$

It then follows from (1) and (2) that

$$\begin{aligned}&2\lambda\sup_{s\in G}\inf_{t\in G}(S(ts)x - Px, Px - Py) \\ &\geq 2\lambda\sup_{s\in G}\inf_{t\in G}(S(tss_3)x - Px, Px - Py) \\ &\geq \sup_{s\in G}\inf_{t\in G}\|\lambda S(tss_3)x + (1-\lambda)Px - Py\|^2 \\ &\quad - \|Px - Py\|^2 - M^2\lambda^2 \\ &= \sup_{s\in G}\inf_{t\in G}\|\lambda S(ts)S(s_3)x + (1-\lambda)PS(s_3)x - Py\|^2 \\ &\quad - \|Px - Py\|^2 - M^2\lambda^2 \\ &\geq \|PS(s_3)x - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2 \\ &= -M^2\lambda^2.\end{aligned}$$

Therefore, we have

$$\sup_{s\in G}\inf_{t\in G}(S(ts)x - Px, Px - Py) \geq -\frac{1}{2}M^2\lambda.$$

Letting $\lambda \to 0$, then we have

$$\sup_{s\in G}\inf_{t\in G}(S(ts)x - Px, Px - Py) \geq 0. \tag{3}$$

Let $\varepsilon > 0$. Then there is $s_4 \in G$ such that

$$k_{ts_4} < \varepsilon$$

for all $t \in G$. For such an $s_4 \in G$, from (3), we have

$$\sup_{s\in G}\inf_{t\in G}(S(ts)S(s_4)x - PS(s_4)x, PS(s_4)x - Py) \geq 0$$

and hence there is $s_5 \in G$ such that

$$\inf_{t\in G}(S(ts_5)S(s_4)x - PS(s_4)x, PS(s_4)x - Py) > -\varepsilon.$$

Then, from $PS(s_4)x = Px$, we have

$$\inf_{t\in G}(S(ts_5s_4)x - Px, Px - Py) > -\varepsilon. \tag{4}$$

Similarly, from (3), we also have

$$\sup_{s\in G}\inf_{t\in G}(S(ts)S(s_5s_4)y - PS(s_5s_4)y, PS(s_5s_4)y - Px) \geq 0,$$

and there exists $s_6 \in G$ such that

$$\inf_{t\in G}(S(ts_6s_5s_4)y - PS(s_5s_4)y, PS(s_5s_4)y - Px) \geq -\varepsilon,$$

that is

$$\inf_{t\in G}(Py - S(ts_6s_5s_4)y, Px - Py) \geq -\varepsilon. \tag{5}$$

On the other hand, from (4)

$$\inf_{t\in G}(S(ts_6s_5s_4)x - Px, Px - Py) > -\varepsilon. \tag{6}$$

Combining (5) and (6), we have

$$\begin{aligned} -2\varepsilon &< (S(ts_6s_5s_4)x - S(ts_6s_5s_4)y, Px - Py) - \|Px - Py\|^2 \\ &\leq \|S(ts_6s_5s_4)x - S(ts_6s_5s_4)y\| \cdot \|Px - Py\| - \|Px - Py\|^2 \\ &\leq (1 + k_{ts_6s_5s_4})\|x - y\| \cdot \|Px - Py\| - \|Px - Py\|^2 \\ &\leq (1 + \varepsilon)\|x - y\| \cdot \|Px - Py\| - \|Px - Py\|^2. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$. This completes the proof.

By using Lemma 2.1 and Theorem 3.1, we have an ergodic retraction theorem for asymptotically nonexpansive semigroups with a nonempty domain which is noconvex and nonclosed.

THEOREM 3.2. *Let C be a nonempty subset of a real Hilbert space H and let $\mathcal{S} = \{S(t) : t \in G\}$ be an asymptotically nonexpansive semigroup on C such that $\mathcal{F}(\mathcal{S}) \neq \emptyset$. Then the following statements are equivalent:*

(1) $\bigcap_{s \in G} \overline{conv}\{S(ts)x : t \in G\} \bigcap \mathcal{F}(\mathcal{S}) \neq \emptyset$ *for each* $x \in C$.
(2) *There is a unique nonexpansive retraction P of C onto $\mathcal{F}(S)$ such that $PS(t) = S(t)P = P$ for every $t \in G$ and $Px \in \overline{conv}\{S(t)x . t \in G\}$ for every $x \in C$*

We denote by $B(G)$ the Banach space of all bounded real valued functions on G with supremum norm. Let X be a subspace of $B(G)$ containing constant functions. Then, a real valued function μ on X is called a *submean on X* if the following conditions are satisfied :

(1) $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$
(3) $f \leq g$ implies $\mu(f) \leq \mu(g)$
(4) $\mu(c) = c$ for every constant c.

The following corollaries are immediately deduced from above theorem.

COROLLARY 3.3. *[7] Let C be a closed convex subset of a Hilbert space H and X a r_s-invariant subspace of $B(G)$ containing constants* *which has a right invariant submean. Let $\mathcal{S} = \{S(t) : t \in G\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^2 \leq 1$ and $\mathcal{F}(\mathcal{S}) \neq \emptyset$, where k_t is the Lipschitzian constant. If for each $x, y \in C$, the function f on G defined by*

$$f(t) = \|S(t)x - y\|^2 \quad \text{for all } t \in G$$

and the function g on G defined by

$$g(t) = k_t^2 \quad \text{for all } t \in G$$

belong to X, then the followings are equivalent:

(1) *$\bigcap_{s \in G} \overline{conv}\{S(ts)x : t \in G\} \bigcap \mathcal{F}(\mathcal{S}) \neq \emptyset$ for each $x \in C$.*

(2) *There is a nonexpansive retraction P of C onto $\mathcal{F}(\mathcal{S})$ such that $PS(t) = S(t)P = P$ for every $t \in G$ and $Px \in \overline{conv}\{S(t)x : t \in G\}$ for every $x \in C$.*

COROLLARY 3.4. *[6] Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(t) : t \in G\}$ be a continuous representation of a semitopological semigroup G as nonexpansive mappings from C into itself. If for each $x \in C$, the set $\bigcap_{s \in G} \overline{conv}\{S(ts)x : t \in G\} \bigcap \mathcal{F}(\mathcal{S}) \neq \emptyset$, then there exists a nonexpansive retraction P of C onto $\mathcal{F}(\mathcal{S})$ such that $PS(t) = S(t)P = P$ for every $t \in G$ and $Px \in \overline{conv}\{S(t)x : t \in G\}$ for every $x \in C$.*

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