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ON THE STUDY OF AFFINE DIFFERENTIAL GEOMETRY OF SURFACE S_2 IN A_4

E. T. Ivlev, O. V. Rozhkova and Hai Gon Je

Abstract In this paper, we investigate the existence of a two dimensional surface in a four dimensional equiaffine space and characterize that surface

1. Introduction

A two-dimensional surface S_2 is viewed in a four-dimensional equiaffine space A_4. We shall mark through L_2 is a tangent plane to S_2 in the current point A, l_1 and l_2 are focus lines of plane L_2; Γ_3^1 and Γ_3^2 are the focal (tangent) 3-planes in meaning [1]; $\rho_1(\rho_2)$ is the characteristic element of 3-plane $\Gamma_3^1(\Gamma_3^2)$ in the direction $l_2(l_1)$ Let's consider points $X \in A_4$ and $X_1 = \mathrm{Pr}_{\Gamma_1^3} X, X_2 = \mathrm{Pr}_{\Gamma_3^2} X$

The totality of all points $X \in A_4$, which are satisfied the point $A \in S_2$, so that corresponding points X_1 and X_2 lie inside corresponding characteristics hyperplanes Γ_3^1 and Γ_3^2, forms a second order hypercone K_2^0 in A_4 with the vertex at the point A

Let Γ_2 be the plane polary associated with the plane L_2 and hypercone K_2^0 : $l_3 = \rho_1 \bigcap \Gamma_2, l_4 = \rho_2 \bigcap \Gamma_2$. Then the plane $P_2 = l_3 \bigcup l_4$ is clothings plane of surface S_2 at the point A : $P_2 \bigcap L_2 = A$, $P_2 \bigcup L_2 = A_4$. In conformity with [2], centre-affinity transformation $\prod(z)$ of the plane L_2 in itself with center A replies of each point $z \in \Gamma_2$. Non-eigen points of the straight lines l_3 and l_4 correspond centre-affinities transformations $\prod_3$ and $\prod_4$, accordingly

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As remarked here affine-invariant geometric images take the possibility to construct the canonical frame of surface S_2 in A_4, with the help which succeed to separate and geometrically to characterize some private classes of surfaces. One of such classes, which is characterized from the following properties:

a) the hypercone K_2^0 on a surface S_2 degenerated in two 3-planes are going through a two-dimensional plane Γ_2,

b) the straight line $l_1(l_2)$ at the centre-affinity transfomation $\prod_3(\prod_4)$ transfers in itself.

It's found that the indicated class of a surface S_2 in A_4 exists and is determined with arbitrariness of six functions of one argument.

2. Invariant rationing of vectors $\vec{e}_3$ and $\vec{e}_4$

The equation of a tangent hyperquadric Q_2 in the local coordinates can be expressed in the form

$$a_{ij}x^i x^j + 2a_{0i}x^i + a_{00} = 0, \tag{2.1}$$

where

$$a_{ij} = (\vec{e}_i * \vec{e}_j),\ a_{0i} = (\vec{r} * \vec{e}_i),\ \vec{a}_{00} = (\vec{r} * \vec{r}). \tag{2.2}$$

A condition for a point to belong to the hyperquadric surface will give

$$a_{00} = (\vec{r} * \vec{r}) = 0. \tag{2.3}$$

A condition for all points to belong to the first differential vicinity (that is a first-order tangency) can be accomplished by differentiating (2.3) and reducing coefficients of independent forms ω^1 and ω^2 to zero.

We obtain $(d\vec{r} * \vec{r}) = 0 \iff \omega^1(\vec{e}_1 * \vec{r}) + \omega^2(\vec{e}_2 * \vec{r}) = 0$. Hence

$$a_{01} \equiv (\vec{r} * \vec{e}_1) = 0,\ a_{02} \equiv (\vec{r} * \vec{e}_2) = 0. \tag{2.4}$$

To support a second-order tangency, one should differentiate (2.4). We obtain

$$(d\vec{r} * \vec{e}_1) + (\vec{r} * d\vec{e}_1) = 0,\ (d\vec{r} * \vec{e}_2) + (\vec{r} * d\vec{e}_2) = 0.$$

Inserting expressions $d\vec{r}$ and $d\vec{e}_\alpha$ with using of (3),(4),(8) and (13) in [4], we can find equating coefficients by ω^1 and ω^2 :

$$(\vec{A} * \vec{e}_3)+(\vec{e}_1 * \vec{e}_1) = 0,\ (\vec{e}_1 * \vec{e}_2) = 0,$$

$$(\vec{A} * \vec{e}_4) + (\vec{e}_2 * \vec{e}_2) = 0 \Longleftrightarrow \tag{2.5}$$

$$a_{03} + a_{11} = 0,\ a_{12} = 0,\ a_{04} + a_{22} = 0. \tag{2.6}$$

To support a third-order tangency, we differentiate (2.5). Taking into consideration

$$3(\vec{e}_1 * \vec{e}_3) + E^*(\vec{A} * \vec{e}_4) = 0, 3(\vec{e}_2 * \vec{e}_4) + E(\vec{A} * \vec{e}_3) = 0,$$

$$(\vec{e}_2 * \vec{e}_3) + (\vec{e}_1 * \vec{e}_1) = 0, (\vec{e}_1 * \vec{e}_4) + (\vec{e}_2 * \vec{e}_2) = 0 \Longleftrightarrow \tag{2.7}$$

$$3a_{13} + E^*a_{04} = 0,\ 3a_{24} + Ea_{03} = 0,$$

$$a_{23} + a_{11} = 0,\ a_{14} + a_{22} = 0. \tag{2.8}$$

On supposing

$$a_{11} = \alpha,\ a_{22} = \alpha^*, \tag{2.9}$$

we can find from (2.6) and (2.7)

$$a_{03} = -\alpha,\ a_{04} = -\alpha^*,\ a_{23} = -\alpha,\ a_{14} = -\alpha^*,$$

$$a_{13} = \frac{E^*\alpha^*}{3},\ a_{24} = \frac{E\alpha}{3}. \tag{2.10}$$

Substituting values of some coefficients a_{ik} found from (3), (4), (8) and (10) in [4], we obtain that all hyperquadrics in A_4, which have a third-oder tangency with the surface S_2, are defined by the equation:

$$\alpha(x^1)^2 + \alpha^*(x^2)^2 - 2\alpha^* x^1 x^4 - 2\alpha x^2 x^3 - 2\alpha x^3 - 2\alpha^* x^4$$

$$+\frac{2}{3}E^*\alpha x^1 x^3 + \frac{2}{3}E\alpha x^2 x^4 + a_{\hat{\alpha}\hat{\beta}} x^{\hat{\alpha}} x^{\hat{\beta}} = 0. \tag{2.11}$$

In view of (32) in [4], it is seen that polars of points t_1 and τ_1 in (2.10) are defined by equations respectively:

$$t_1 : \ \alpha x^1 + \frac{1}{3}E^*\alpha^* x^3 - \alpha^* x^4 = 0,$$

$$\tau_1 : \ \alpha^* x^2 + \frac{1}{3}E\alpha x^4 - \alpha x^3 = 0. \tag{2.12}$$

If this system is considered regarding to α and α^*, then we can obtain that it will have non-trivial solutions according to α and α^* if and only if

$$Q_2 : \ x^1 x^2 - (1 + \frac{\overline{EE^*}}{9})x^3 x^4 + \frac{E^*}{3}(x^3)^2 + \frac{E}{3}(x^4)^2 = 0. \tag{2.13}$$

We call Q_2 the aggregate of all points (36) in [4] in A_4, to each of them corresponds the aggregate of such hyperquadric (2.10), according to which points $\vec{t}_1$ and $\vec{\tau}_1$ have the same polar. It follows from (2.11) and (2.12) that Q_2 is the hyperquadric in A_4, defined by equation (2.13).

Points with radius vectors

$$\vec{t}_1{}^* = \vec{A} + \vec{e}_1,$$

$$\vec{\tau}_1{}^* = \vec{A} + \vec{e}_2,$$

which are symmetrical to points (32) in [4] on the corresponding straight lines, are taken up. The point with the radius vector

$$V = \vec{A} + \frac{1}{2}(\vec{e}_1 + \vec{e}_2)$$

is the middle of the segment $[\vec{t}_1{}^*, \vec{\tau}_1{}^*]$ In view of (17) in [4], it is seen that the curve

$$K . \ \omega^2 = \omega^1, \ \omega^{\hat{\alpha}} = 0$$

on the surface S_2 is geometrically characterized, because the point $\vec{A}$ describes a line with the tangent along the curve, which parallels to the straight line $A_V = (\vec{A}, \vec{e}_1 + \vec{e}_2)$. From

$$\begin{aligned} d(\vec{e}_1 + \vec{e}_2) &= (...)^1\vec{e}_1 + (...)^2\vec{e}_2 + \omega_1^3\vec{e}_3 + \omega_2^4\vec{e}_4 \\ &= (...)^1\vec{e}_1 + (...)^2\vec{e}_2 + \omega^1\vec{e}_3 + \omega^2\vec{e}_4, \end{aligned} \tag{2.14}$$

we notice that the straight line $A_{V^*} = (\vec{A}, \vec{e}_3 + \vec{e}_4)$ is the intersection of the plane $\Gamma_2 = (\vec{A}, \vec{e}_3, \vec{e}_4)$ with 3-dimensional plane passing through $L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2)$ and the tangent linear subspace to the aggregate of straight lines A_V along the curve K.

Let us consider the point on the straight line $l_3 = (\vec{A}, \vec{e}_3)$

$$\vec{T}_3 = \vec{A} + t\vec{e}_3,$$

which is in direction A_{V^*} projected at the point $\vec{T}_4 = \vec{A} + t\vec{e}_4$ on the straight line $l_4 = (\vec{A}, \vec{e}_4)$

Let points T_3 and T_4 be such points that $(\vec{t}_1, \vec{\tau}_1, \vec{T}_3, \vec{T}_4) = 1$, then $t^2 = 1$. Consequently, on lines l_3 and l_4 points

$$\vec{\varepsilon}_3 = \vec{A} + \vec{e}_3, \ \vec{\varepsilon}_3^{\,*} = \vec{A} - \vec{e}_3,$$

$$\vec{\varepsilon}_4 = \vec{A} + \vec{e}_4, \ \vec{\varepsilon}_4^{\,*} = \vec{A} - \vec{e}_4,$$

give the geometrical meaning of rationing of vectors $\vec{e}_3$ and $\vec{e}_4$. It follows from (2.13) that the hyperplane $\Gamma_2 = (\vec{A}, \vec{e}_3, \vec{e}_4)$ and the hyperquadric Q_2 intersect in two straight lines:

$$\vec{u}_4 = (\vec{A}, E^*\vec{e}_4 + 3\vec{e}_3), \ \vec{u}_3 = (\vec{A}, E\vec{e}_3 + 3\vec{e}_4).$$

Hence, invariants E and E^* are geometrically characterized in following manner: $E = 3\omega, \ E^* = 3\omega^*$

Here formulas

$$\omega = \{(\vec{A}, \vec{e}_3), \vec{u}_3; (\vec{A}, \vec{e}_3 + \vec{e}_4); (\vec{A}, \vec{e}_4)\},$$

$$\omega^* = \{(\vec{A}, \vec{e}_3), (\vec{A}, \vec{e}_3 + \vec{e}_4); \vec{u}_4; (\vec{A}, \vec{e}_4)\},$$

are complex connections of the corresponding four straight lines passing through the point $\vec{A} \in S_2$ in the plane Γ_2

3. Some affine-invariant geometrical images

For geometrical interpretation of some special classes of the surface S_2 in A_4, which are to be discussed in the next section, in this section let us consider some affine-invariant geometrical images associated with the surface S_2 in A_4 We shall conduct a research of these images, using terms of the canonical frame built analytically in [4] and geometrically in the preceding section.

3.1. The diversity $\{L_2, \Gamma_2\}$ is a two dimensional diversity of pairs of planes L_2 and Γ_2

3.1.1. Some affinities of the tangent plane L_2.

Let us take up the point in Γ_2: $\vec{Z} = \vec{A} + z^{\hat{\alpha}}\vec{e}_{\hat{\alpha}} \in \Gamma_2$.

We have: $d\vec{Z} = (...)^{\hat{\alpha}}\vec{e}_{\hat{\alpha}} + x^{\hat{\alpha}}A^{\alpha}_{\hat{\alpha}\beta}\omega^{\beta}\vec{e}_{\alpha}$.

Therefore, to each point $\vec{Z} \in \Gamma_2$ corresponds the centre-affine into-transformation of the plane L_2 with the vector $\vec{A}$ ((7) in [2]):

$$\Pi(z) = \{\delta^{\alpha}_{\beta} + z^{\hat{\alpha}}A^{\alpha}_{\hat{\alpha}\beta}\}. \tag{3.1}$$

This affinor transfers each direction

$$x = (\vec{A}, \vec{e}_{\beta})x^{\beta} \in L_2 \tag{3.2}$$

to the following direction

$$y = (\vec{A}, \vec{e}_{\alpha})y^{\alpha} \in L_2, \; y = (z)x,$$
$$y^{\alpha} = \{\delta^{\alpha}_{\beta} + z^{\hat{\alpha}}A^{\alpha}_{\hat{\alpha}\beta}\}x^{\beta}, \tag{3.3}$$

thus $y = L_2 \bigcap \{\Gamma_2 \bigcap T(z, x)\}$.

Here, $T(z, x)$ means the line described by the point $\vec{Z} \in \Gamma_2$ in the direction of x. It follows from (15) and (41) in [4] that there are two invariant affinors Π_3 and Π_4 of the plane L_2, which are the affinor $\Pi(z)$, responding with non-eigen points of straight lines l_3 and l_4

$$\Pi_3 = \{A^{\alpha}_{3\beta}\}, \; \Pi_4 = \{A^{\alpha}_{4\beta}\}. \tag{3.4}$$

We shall put the following geometrical images.

1.) The straight line $l^* = \{Z \in \Gamma_2 | \; ter\Pi(z) = 0\}$,

2.) The conic $\psi^1_1 = \{Z \in \Gamma_2 | \; ter\Pi^2(z) = 0\}$,

3.) The focus conic $\psi^2_1 = \{Z \in \Gamma_2 | \; det\Pi(z) = 0\}$.

It follows from (3.1) that each of these geometrical images in Γ_2 is defined by equations respectively:

$$l^*_1 : \; 1 + 2a_{0\hat{\alpha}}z^{\hat{\alpha}} = 0, \; z^{\alpha} = 0, \tag{3.5}$$

$$\psi_1^1: \; 1 + 2a_{0\hat{\alpha}} z^{\hat{\alpha}} + a_{\hat{\alpha}\hat{\beta}} z^{\hat{\alpha}} z^{\hat{\beta}} = 0, \; z^{\alpha} = 0, \tag{3.6}$$

$$\psi_1^2: \; 1 + 2a_{0\hat{\alpha}} z^{\hat{\alpha}} + b_{\hat{\alpha}\hat{\beta}} z^{\hat{\alpha}} z^{\hat{\beta}} = 0, \; z^{\alpha} = 0, \tag{3.7}$$

where

$$\begin{aligned} a_{0\hat{\alpha}} &= \frac{1}{2} A^{\alpha}_{\hat{\alpha}\alpha}, \; a_{\hat{\alpha}\hat{\beta}} = \frac{1}{2} A^{\alpha}_{\hat{\alpha}\beta} A^{\beta}_{\hat{\beta}\alpha}, \\ b_{\hat{\alpha}\hat{\beta}} &= \frac{1}{2}(A^1_{\hat{\alpha}1} A^1_{\hat{\beta}1} + A^2_{\hat{\alpha}2} A^2_{\hat{\beta}2} - A^1_{\hat{\alpha}2} A^2_{\hat{\beta}1} - A^2_{\hat{\alpha}1} A^1_{\hat{\beta}2}). \end{aligned} \tag{3.8}$$

It follows from (3.5)–(3.7) that the straight line l^* is a polar of the point $\vec{A}$ in the conic ψ_1^1 or ψ_1^2. Thus, to each point $\vec{Z} \in \Gamma_2$ correspond centre–affinities Π_3 and Π_4

3.1.2. Affine connections C_{12} and C_{34}

1). By analogy with [2] we shall consider the connection C_{12}, which is the mapping of the adjoining plane L_2' onto the initial L_2 in the direction of plane Γ_2.

This mapping is defined by forms ω^{α} and ω^{β}_{α}, which, by virtue of (2) and (3) in [4], satisfy structural equations

$$D\omega^{\alpha} = \omega^{\beta} \wedge \omega^{\alpha}_{\beta},$$

$$D\omega^{\beta}_{\alpha} = \omega^{\gamma}_{\alpha} \wedge \omega^{\beta}_{\gamma} + R^{\beta}_{\alpha 12} \omega^1 \wedge \omega^2,$$

where curvature tensor components are defined by formulas, by virtue of (8) and (14) in [4]:

$$\begin{aligned} R^1_{112} &= \frac{1}{2} A^1_{32}, \; R^2_{212} = -\frac{1}{2} A^2_{41}, \\ R^2_{112} &= \frac{1}{2} A^2_{32}, \; R^1_{212} = -\frac{1}{2} A^1_{41}. \end{aligned} \tag{3.9}$$

We shall call

$$R_1 = \{R^{\beta}_{\alpha 12}\} \tag{3.10}$$

the affine into–transformation of curvature of the plane L_2 in the meaning [3].

2). The connection C_{34} is the mapping of the adjoining plane Γ^1_2 onto the initial Γ_2 in the direction of L_2 [2].

This mapping is defined by forms $\omega^{\hat\beta}_{\hat\alpha}$, which, by virtue of (2) and (3) in [4], satisfy structural equations

$$D\omega^{\hat\beta}_{\hat\alpha} = \omega^{\hat\gamma}_{\hat\alpha} \wedge \omega^{\hat\beta}_{\hat\gamma} + R^{\hat\beta}_{\hat\alpha 12}\omega^1 \wedge \omega^2,$$

where curvature tensor components are defined, by virtue of (8) and (14) in [4], in formulas:

$$\begin{gathered} R^3_{312} = -\frac{1}{2}A^1_{32},\ R^4_{412} = \frac{1}{2}A^2_{41}, \\ R^4_{312} = \frac{1}{2}A^2_{31},\ R^3_{412} = -\frac{1}{2}A^1_{42}. \end{gathered} \tag{3.11}$$

We shall call

$$R_2 = \{R^{\hat\beta}_{\hat\alpha 12}\} \tag{3.12}$$

the into–affinor of curvature of the plane Γ_2 in the meaning [3]

3.2. The diversity $\{\rho_1, \rho_2\}$ is a two–dimensional diversity of pairs of planes $\rho_1 = (\vec{A}, \vec{e}_1, \vec{e}_3)$ and $\rho_2 = (\vec{A}, \vec{e}_2, \vec{e}_4)$

1). The connection C_{13} is the mapping of the adjoining plane ρ_1' onto the initial plane ρ_1 in the direction of ρ_2. This mapping is defined by forms ω^b_a $(a, b, c = 1, 3,\ \hat a, \hat b = 2, 4)$, which satisfy structural equations

$$D\omega^a = \omega^b \wedge \omega^a_b + \breve{R}^a_{012}\omega^1 \wedge \omega^2.$$

$$D\omega_a^b = \omega_a^c \wedge \omega_b^c + \breve{R}_{a12}^b \omega^1 \wedge \omega^2,$$

where torsion curvature tensor components are defined by formulas:

$$\breve{R}_{012}^3 = 0,\ \breve{R}_{012}^1 = -\frac{1}{2},\ \breve{R}_{112}^1 = -\frac{1}{2},\ \breve{R}_{312}^3 = \frac{EE^*}{2},$$
$$\breve{R}_{112}^3 = 0,\ \breve{R}_{312}^1 = -\frac{1}{2}(A_{32}^2 - E^* A_{42}^1). \tag{3.13}$$

we shall call

$$\breve{R}_1 = \{\breve{R}_{012}^\beta, \breve{R}_{012}^b\} \tag{3.14}$$

the affine into–transformation of curvature of the plane ρ_1.

2). The connection C_{24} is the mapping of the adjoining plane ρ_2' onto the initial plane ρ_2 in the direction of ρ_1.

This mapping is defined by forms $\omega_{\hat{a}}^{\hat{b}}$, which satisfy structural equations

$$D\omega^{\hat{a}} = \omega^{\hat{b}} \wedge \omega_{\hat{b}}^{\hat{a}} + \breve{R}_{012}^{\hat{a}} \omega^1 \wedge \omega^2,$$

$$D\omega_{\hat{a}}^{\hat{b}} = \omega_{\hat{a}}^{\hat{c}} \wedge \omega_{\hat{b}}^{\hat{c}} + \breve{R}_{\hat{a}12}^{\hat{b}} \omega^1 \wedge \omega^2,$$

where torsion curvature tensor components are defined in formulas :

$$\breve{R}_{012}^4 = 0,\ \breve{R}_{012}^2 = \frac{1}{2},\ \breve{R}_{212}^2 = \frac{1}{2},\ \breve{R}_{412}^4 = -\frac{EE^*}{2},$$
$$\breve{R}_{212}^4 = 0,\ \breve{R}_{412}^2 = \frac{1}{2}(A_{41}^1 - E A_{31}^2). \tag{3.15}$$

We shall call

$$\breve{R}_2 = \{\breve{R}_{012}^{\hat{a}}, \breve{R}_{\hat{a}12}^{\hat{b}}\} \tag{3.16}$$

the affine into–transformation of curvature of the plane ρ_2.

3.3. The diversity $\{\rho_1^*, \rho_2^*\}$ is the two– dimensional diversity of pairs of planes $\rho_1^* = (\vec{A}, \vec{e}_1, \vec{e}_4)$ and $\rho_2^* = (\vec{A}, \vec{e}_2, \vec{e}_3)$

3.3.1. Affine connections C_{14} and C_{23}

1). The connection C_{14} is the mapping of the adjoing plane $\rho_1^{*'}$ onto the initial plane ρ_1^* in the direction of ρ_2^*. This mapping is defined by forms ω_p^q $(p, q, r = 1, 4;\ \hat{p}, \hat{q}, \hat{r} = 2, 3)$, which satisfy structural equations:

$$D\omega^p = \omega^q \wedge \omega_q^p + \tilde{R}^p_{012}\omega^1 \wedge \omega^2,$$

$$D\omega_p^q = \omega_p^r \wedge \omega_r^q + \tilde{R}^q_{p12}\omega^1 \wedge \omega^2,$$

where torsion curvature tensor components are defined in formulas :

$$\tilde{R}^4_{012} = 0,\ \tilde{R}^1_{012} = -\frac{1}{2},\ \tilde{R}^1_{112} = -\frac{1}{2}(1 - A^1_{32}),\ \tilde{R}^4_{112} = 0;$$

$$\tilde{R}^4_{412} = \frac{1}{2}(A^2_{41} - EE^*),\ \tilde{R}^1_{412} = -\frac{1}{2}(A^2_{42} + EA^1_{31}). \tag{3.17}$$

We shall call

$$\tilde{R}_1 = \{\tilde{R}^p_{012}, \tilde{R}^q_{p12}\} \tag{3.18}$$

the (linear) affine into-transformation of the plane ρ_1^*.

2). The connection C_{23} is the mapping of the adjoning plane $\rho_2^{*'}$ onto the initial plane ρ_2^* in the direction of ρ_1^*. This mapping is defined by forms $\omega_{\hat{p}}^{\hat{q}}$, which satisfy structural equations

$$D\omega^{\hat{p}} = \omega^{\hat{q}} \wedge \omega_{\hat{q}}^{\hat{p}} + \tilde{R}^{\hat{p}}_{012}\omega^1 \wedge \omega^2,$$

$$D\omega_{\hat{p}}^{\hat{q}} = \omega_{\hat{p}}^{\hat{r}} \wedge \omega_{\hat{r}}^{\hat{q}} + \tilde{R}^{\hat{q}}_{\hat{p}12}\omega^1 \wedge \omega^2,$$

where torsion curvature tensor components are defined in the following formulas :

$$\tilde{R}^3_{012} = 0,\ \tilde{R}^2_{012} = \frac{1}{2},\ \tilde{R}^2_{212} = \frac{1}{2}(1 - A^2_{41}),\ \tilde{R}^3_{212} = 0,$$

(3.19)

$$\tilde{R}^3_{312} = -\frac{1}{2}(A^1_{32} - EE^*),\ \tilde{R}^2_{312} = \frac{1}{2}(A^1_{31} + E^* A^2_{42}).$$

We shall call

$$\tilde{R}_2 = \{\tilde{R}^{\hat{p}}_{012}, \tilde{R}^{\hat{q}}_{\hat{p}12}\} \tag{3.20}$$

the (linear) affine into–trasformation of the plane ρ_2^*.

3.3.2. Focus conics φ_1^* and φ_2^* of the planes ρ_1^* and ρ_2^*

Focus conics φ_1^* and φ_2^* of the planes ρ_1^* and ρ_2^* are defined by the equations

$$\begin{aligned}&\varphi_1^* : \ (x^1)^2 + x^1 + A^2_{42}x^1x^4 - A^2_{41}E(x^4)^2 = 0, \ x^2 = 0, \ x^3 = 0,\\ &\varphi_2^* : \ (x^2)^2 + x^2 + A^1_{31}x^2x^3 - A^1_{32}E^*(x^3)^2 = 0, \ x^1 = 0, \ x^4 = 0.\end{aligned} \tag{3.21}$$

The centres of these conics are points:

$$\begin{aligned}\bar{V}_{14} &= \vec{A} - \frac{2EA^1_{41}}{4EA^1_{41} + (A^2_{42})^2}\vec{e}_1 - \frac{A^2_{42}}{4EA^1_{41} + (A^2_{42})^2}\vec{e}_4,\\ \bar{V}_{23} &= \vec{A} - \frac{2E^*A^1_{32}}{4E^*A^1_{32} + (A^1_{31})^2}\vec{e}_2 - \frac{A^1_{31}}{4E^*A^1_{32} + (A^1_{31})^2}\vec{e}_3.\end{aligned} \tag{3.22}$$

4. Invariant classes of the two dimensional surfaces S_2 in A_4

With equiaffine-invariant geometrical images and connections taken up in the preceding items let analytically characterize invariant classes of the two-dimensional surfaces in A_4. We point out some of them:

1). Consider the class

$$E = 0, \ E^* = 0. \tag{4.1}$$

In view of (13) in [4], it is seen that

$$\omega_3^4 = 0, \ \omega_4^3 = 0.$$

Differentiating equations externally, we have convinced that along the surface of class (4.1) in (13) from [4] correlations

$$A^2_{31} = 0, \ A^1_{42} = 0 \tag{4.2}$$

are accomplished.

THEOREM 1. *The surface S_2 in A_4 of class (4.1) is simultaneously characterized by the following properties.*

a) The conic φ_1^* in the plane $(\vec{A}, \vec{e}_1, \vec{e}_4)$ disintegrates into two straight lines

$$(\vec{A}, \vec{e}_4) \quad x^1 + A^2_{42}x^4 + 1 = 0, \; x^2 = 0, \; x^3 = 0,$$

b) The conic φ_2^* in the plane $(\vec{A}, \vec{e}_2, \vec{e}_3)$ disintegrates into two straight lines

$$(\vec{A}, \vec{e}_3) \quad x^2 + A^1_{31}x^3 + 1 = 0, \; x^1 = 0, \; x^4 = 0.$$

The proof of this theorem is immediately from (35), (18), (26) and (38) in [4] with making allowance for (4.1).

From (4 2), taking into consideration (3.17), (3.24), (3.26)–(3.34), we conclude that the surface S_2 in A_4 of class (4.1) has the following properties:

a) The straight line $(\vec{A}, \vec{e}_1)$ under the affinity Π_3 transfers into the line $(\vec{A}, \vec{e}_1)$ and the straight line $(\vec{A}, \vec{e}_2)$ under the affinity Π_3 transfers into the line $(\vec{A}, \vec{e}_2)$.

b) Vectors $\vec{e}_3$ and $\vec{e}_4$ are main directions under the affinity R_2.

c) The plane ρ_1 under the affinity $\breve{R}_1$ transfers into the straight line, which parallels to the straight line $(\vec{A}, \vec{e}_1 + A^2_{32}\vec{e}_3)$,

and the straight line $(\vec{A}, \vec{e}_2 + A^1_{41}\vec{e}_4)$ parallels to an image of the plane ρ_2 under the affinity $\breve{R}_2$.

d) The hypercone K^0_2 disintegrates into two hyperplanes L^1_3 and L^2_3.

THEOREM 2. *The surface S_2 in A_4 of class (4.1) exists and is defined with the arbitrariness of six functions of one argument.*

PROOF. From (15) and (16) in [4] and by virtue of (4.1) and (4.2), we obtain

$$A^2_{22} + A^1_{41} = 1,\ A^1_{11} + A^2_{32} = 1,\ A^3_{32} = 2A^1_{12} - 1,\ A^4_{41} = 2A^2_{21} - 1,$$
$$-3A^1_{32} - A^2_{21} + 3 = 0,\ -3A^2_{41} + 3 - A^1_{12} = 0\ ,$$
$$dA^1_{31} \wedge \omega^1 + dA^1_{32} \wedge \omega^2 =$$
$$(2A^1_{31} - A^2_{32} - 2A^2_{21}A^1_{32} - 2A^1_{11}A^1_{32} - 2A^1_{12}A^1_{31})\omega^1 \wedge \omega^2,$$
$$dA^2_{41} \wedge \omega^1 + dA^2_{42} \wedge \omega^2 =$$
$$(2A^2_{42} - A^1_{41} - 2A^1_{12}A^2_{41} - 2A^2_{22}A^2_{41} - 2A^2_{21}A^2_{42})\omega^2 \wedge \omega^1,$$
$$dA^2_{32} \wedge \omega^2 = (1 - 2A^1_{12} + 2A^1_{12}A^2_{21} + A^2_{32}(1 - A^1_{11} - 4A^2_{21})$$
$$- A^2_{21} + A^1_{31})\omega^1 \wedge \omega^2,$$
$$dA^1_{41} \wedge \omega^1 = (A^2_{42} - A^1_{41}(3A^1_{12} + A^2_{22}))\omega^2 \wedge \omega^1,$$
$$dA^1_{11} \wedge \omega^1 + dA^1_{12} \wedge \omega^2 =$$
$$(-1 + A^1_{32} + A^1_{11} - A^1_{12} - A^1_{11}A^1_{12} + A^2_{21}A^1_{12}),$$
$$dA^2_{21} \wedge \omega^1 + dA^2_{22} \wedge \omega^2 =$$
$$(-1 + A^2_{22} - A^2_{21} + A^2_{41} - A^2_{22}A^2_{21} + A^2_{21}A^1_{12})\omega^2 \wedge \omega^1.$$

Applying Bachvalov's theorem to the above system , we obtain

$$r = 10 - 4 = 6,\ s_1 = 6 \Longrightarrow r = s_1 = 6.$$

Thus, the arbitrariness of the solution is equal to six functions of one argument.

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E. T. Ivlev,O. V. Rozhkova
Department of Higher Mathematics
Tomsk Polytechnic University
Tomsk,634034, Russia

Hai Gon Je
Department of Mathematics
University of Ulsan
Ulsan 680–749, Korea