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ON INTEGRATION OF VECTOR-VALUED FUNCTIONS FOR AN OPERATOR-VALUED MEASURE

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1. Introduction

The purpose of this paper is to present an integration theory for the case of vector-valued functions with respect to an operator-valued measure defined on a σ-algebra of Borel subsets of S with values in $L(X, Y)$, the space of all continuous linear operators from a locally convex space X into a locally convex space Y equipped with topology of bounded convergence It is well-known that a traditional integration theory is to define the integral of a simple function and then extend the integral by some limit process to a general case of functions in [1], [2].

The idea of this type of integration has been introduced by several authors in [2], [3], [4] and [7]. In these papers, either the integrands or the integrals or both have their values in Banach spaces and in [7] the author considered the integration of scalar-valued functions with respect to operator-valued measures

In this paper we consider an integration theory for vector-valued functions with respect to finitely additive measures similar to ones in [2], [3], and then generalize [4], [7] in a locally convex space setting, using a weak approach.

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2. Preliminaries

We will denote by $B(S)$ the σ-algebra of Borel subsets of a compact Hausdorff space S, X and Y denote two locally convex Hausdorff spaces, being Y complete, and Q generating families of continuous seminorms on Y. Let X' and Y' be the topological duals of X and Y respectively.

An operator-valued measure $\mu : B(S) \to L(X,Y)$ is an additive set function with

$$\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n),\ \{E_n\} \subset B(S) \text{ with } \cup_{n=1}^{\infty} E_n \in B(S).$$

$E_i \cap E_j = \emptyset (i \neq j), i, j = 1, 2, \ldots$, the series being unconditionally convergent with respect to the topology of simple convergence.

Let us suppose that there exists a vector measure $\nu : B(S) \to X$ and let μ be a non-negative real-valued measure on $B(S)$. If $\lim_{\mu(E)\to 0} \nu(E) = 0$, then ν is called μ-continuous and this denoted by $\nu \ll \mu$. When $\nu \ll \mu$, sometimes μ is said to be a control measure for ν.

It is well-known that if $\mu : B(S) \to L(X,Y)$ is an operator-valued measure, then for each $x \in X$, the set function $\mu_x : B(S) \to Y$, defined by $\mu_x(E) = \mu(E)x$ is a vector measure and conversely, if $\mu(\ )x$ is a vector measure, then $\mu : B(S) \to L(X,Y)$ is countably additive with respect to the topology of simple convergence in $L(X,Y)$. From the above result it can be proved that the set function $y'\mu : B(S) \to X'$ defined by $(y'\mu)(E)x = y'(\mu(E)x)$ for each $E \in B(S)$ is an X'-valued measure. If $y' \in Y'$ and $q \in Q$, we will write $y' \leq q$ whenever $|y'(y)| \leq q(y)$ for $y \in Y$.

DEFINITION 2.1. We define the *q-variation of* μ, which is a finite set function on $B(S)$, as

$$|\mu|_q(E) = \sup \sum_{i=1}^{n} q(\mu(E \cap E_i)),\ E \in B(S),$$

where the supremum is taken over all finite pairwise disjoint sets $\{E_i\} \subset B(S)$. For each $y' \in Y'$, the variation of $y'\mu$, as

$$|y'\mu|(E) = \sup \sum_{i=1}^{n} |y'\mu(E \cap E_i)|.$$

DEFINITION 2.2. We define the *q-semi-variation of* μ as

$$\| \widehat{\mu} \|_q (E) = \sup_{y' \leq q} |y'\mu|_q(E), \; E \in B(S),$$

which is non-negative and not necessarily finite. Note that $\| \widehat{\mu} \|_q (E) < \infty$ whenever $|y'\mu|_q(E) < \infty$ for each $y' \in Y'$.

From the above Definition,

$$\| \widehat{\mu} \|_q (E) \leq 2 \sup_{F \subset E} q(\mu(F)) \text{ for } E \in B(S), \; q \in Q.$$

It is proved easily that $\| \widehat{\mu} \|_q (\cdot)$ is monotone, subadditive and that

$$|\mu|_q(E) \leq \| \widehat{\mu} \|_q (E) \leq 4 \sup_{y' \leq q} \sup_{F \subset E} |y'\mu(F)|.$$

DEFINITION 2.3[1] A function $f : S \to X$ is said to be *μ-measurable* if there exists a sequence (f_n) of simple functions converging μ-a.e to f. We say that $E \in B(S)$ is a μ-null set if $|y'\mu|_q(E) = 0$ for every $y' \in Y'$, i.e , if $\mu(F) = 0$ for every $F \in B(S)$, $F \subset E$.

3. Integration with respect to an operator-valued measure

Let $\mu : B(S) \to L(X, Y)$ be an operator-valued measure with $|y'\mu|_q(E) < \infty$ for each $E \in B(S)$ and $y' \in Y'$. Also the integrands are assumed to be measurable.

If $E \subset S$, then χ_E will denote its characteristic function on S By a simple function f on S with values in X, we mean a function of the form

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

where $x_i \in X$, $E_i \in B(S)$ and $E_i \cap E_j = \emptyset$ $(i \neq j)$, $i, j = 1, 2, \ldots, n$. we define as usual

$$\int_E f(s)d\mu(s) = \int_E f d\mu = \sum_{i=1}^{n} \mu(E \cap E_i)x_i \in Y, \ E \in B(S)$$

According to [7] we shall define the concept of μ-integrability.

DEFINITION 3.1. A function $f : S \to X$ is said to be a *weakly μ-integrable* if the following conditions hold

(i) f is $y'\mu$ -integrable in the sense of [2], [3], and

(ii) For every $E \in B(S)$ there exists $y_E \in Y$ such that $y'(y_E) = \int_E f dy'\mu$ for every $y' \in Y'$.

If f is μ-integrable, we will denote $y_E = \int_E f d\mu$.

It follows from the above definition if $f : S \to X$ is $y'\mu$-integrable, then $\| f \|$ is $\| y'\mu \|$ $(\cdot)$-integrable and

$$\left| \int_E f dy'\mu \right| \leq \int_E |f| d|y'\mu| \ \text{ for each } E \in B(S) \text{ and if } f : S \to X$$

is a bounded μ-integrable function, then for each $E \in B(S)$,

$$q(\int_E f d\mu) \leq \| f \|_S \| \widehat{\mu} \|_q (E), \ \text{ where } \ \| f \|_S = \sup_{s \in S} |f(s)|$$

PROPOSITION 3.2. *For each μ-integrable function f, given $\epsilon > 0$, there exists a simple function g such that $0 \leq g \leq f$,*

$$\sup_{y' \leq q} \int_S |f - g| d|y'\mu| < \epsilon.$$

PROOF. Since g is a simple function and f is integrable, choose $E \in B(S)$ such that $E = \{|f - g| > \frac{\epsilon}{2(1+\|\widehat{\mu}\|_q(S))}\}$. Then there exists $M > 0$ such that $\| \widehat{\mu} \|_q (E) < \frac{\epsilon}{4M}$ and $q(\int_{S \cap E^c} (f - g) d\mu) \leq \frac{\epsilon}{2(1+\|\widehat{\mu}\|_q(S))}$

$\times \parallel \widehat{\mu} \parallel_q (S \cap E^c) < \frac{\epsilon}{2}$. Also since f and g are bounded, $|g| \leq |f| \leq M$ μ-a.e., there exists $F \in B(S)$ such that $S \cap F^c$ is μ-null and $|f-g| \leq 2M$ for all $s \in F$ Thus

$$\begin{aligned}\sup_{y' \leq q} |\int_{S\cap E} |f-g| d|y'\mu| &\leq \sup_{y' \leq q} |\int_{S\cap E\cap F} |f-g| d|y'\mu| \\ &\quad + \sup_{y' \leq q} |\int_{S\cap E\cap F^c} |f-g| d|y'\mu| \\ &= \sup_{y' \leq q} |\int_{S\cap E\cap F} |f-g| d|y'\mu| + 0 \ \ (\because (S\cap E)\cap F^c \text{ is } \mu\text{-null}) \\ &\leq 2M \parallel \widehat{\mu} \parallel_q (S\cap E\cap F) \\ &\leq 2M \parallel \widehat{\mu} \parallel_q (E) < \frac{\epsilon}{2}.\end{aligned}$$

Hence

$$\sup_{y' \leq q} \int_S |f-g| d|y'\mu| < \epsilon$$

and the proof is complete

PROPOSITION 3 3 *If f $S \to X$ is $|y'\mu|$-integrable and we define $\parallel f \parallel_\mu = \sup_{y' \leq q} \int_E |f| d|y'\mu|$, then $\parallel \cdot \parallel_\mu$ is positively homogeneous and subadditive*

PROOF. Let f and g be $y'\mu$-integrable and $\alpha > 0$. Then

$$\begin{aligned}\parallel \alpha f \parallel_\mu &= \sup_{y' \leq q} \int_E |\alpha f| d|y'\mu| = \alpha \sup_{y' \leq q} \int_E |f| d|y'\mu| \\ &= \alpha \parallel f \parallel_\mu .\end{aligned}$$

Furthermore, since for all $q \in Q$, $y' \in Y'$,

$$\int_E |f+g| d|y'\mu| \leq \int_E |f| d|y'\mu| + \int_E |g| d|y'\mu|.$$

Then

$$\parallel f+g \parallel_\mu \leq \parallel f \parallel_\mu + \parallel g \parallel_\mu$$

Theorem 3.4. *Let $f : S \to X$ be a μ-integrable. Then we have the following:*

(i) $\nu(E) = \int_E f d\mu$ *is bounded with respect to a finitely additive measure* μ.

(ii) $\nu \ll \mu$.

(iii) *there exists an increasing sequence* $\{S_n\}$ *in* $B(S)$ *such that* $\|\widehat{\nu}\|_q(S \cap S_n{}^c) < \frac{1}{n}$ *and* $\|\widehat{\nu}\|_q(S \cap F_n{}^c) = 0$, *where* $F_n = \cup_{k=n}^{\infty} S_k$.

Proof. It is obvious that $\nu(E) = \int_E f d\mu$ is a measure on $B(S)$. Assume that f is bounded on $E \in B(S)$, i.e., for $M > 0$, $|f| \leq M$. By assumption, for every $q \in Q$ there exists $\alpha > 0$ such that, if $U = \{y \in Y : q(y) \leq 1\}, R(\mu) \subset \alpha U$, that is, $|\mu|_q(E) \leq \alpha$ for every $E \in B(S)$. Since for all $E \in B(S)$ there exists $y' \leq q$ such that $|\int_E f dy'\mu| = q(\int_E f d\mu)$, then, from the fact that

$$\sup_{y' \leq q} |y' \int_E f d\mu| \leq \sup_{y' \leq q} \int_E |f| d|y'\mu| \leq M \sup_{y' \leq q} |y'\mu|(E) \leq 2M \cdot \alpha,$$

it follows that $q(\int_E f d\mu) \leq 2M\alpha$, that is, $R(\nu) \subset M\alpha U$. For arbitrary f, from Proposition 3.2, for every $\epsilon > 0$ there exists a simple function g such that

$$\sup_{y' \leq q} \int_E |g - f| d|y'\mu| < \epsilon \text{ for } y' \in Y'.$$

Thus for every $q \in Q$,

$$\begin{aligned} \sup_{y' \leq q} |y' \int_E f d\mu| &\leq \sup_{y' \leq q} \int_E |f| d|y'\mu| \qquad (*) \\ &\leq \sup_{y' \leq q} [\int_E |f - g| d|y'\mu| + \int_E |g| d|y'\mu|] \\ &\leq \epsilon \parallel \widehat{\mu} \parallel_q (E) + \int_E |g| d|y'\mu|. \end{aligned}$$

But since g is bounded, there is $\beta(> 0)$ such that $\{\int_E g d\mu,\ E \in B(S)\} \subset$

βU, thus

$$\sup_{y'\leq q}\left|y'\int_E f\,d\mu\right| \leq 2\alpha\epsilon + 2\beta,$$

$$\text{that is,}\quad \left\{\int_E f\,d\mu,\ E\in B(S)\right\}\subset (2\epsilon\alpha+2\beta)U.$$

To prove (ii) note that, if f is bounded, say $|f|\leq M$, then

$$\begin{aligned}\sup_{y'\leq q}|y'\int_E f\,d\mu| &\leq \sup_{y'\leq q}\int_E |f|d|y'\mu| \leq M\sup_{y'\leq q}|y'\mu|(E)\\ &\leq M\parallel\widehat{\mu}\parallel_q(E)\end{aligned}$$

and, therefore, for arbitrary f, applying for all $\epsilon>0$, Theorem 3.2, and taking $|g|\leq M'$, it follows as in the proof of $(*)$

$$\sup_{y'\leq q}|y'\int_E f\,d\mu| \leq \epsilon\parallel\widehat{\mu}\parallel_q(S) + M'\parallel\widehat{\mu}\parallel_q(E),$$

whence, for a suitable $\delta>0$, if $\parallel\widehat{\mu}\parallel_q(E)<\delta$ then $q(\int_E f\,d\mu)<\epsilon$, for every $q\in Q$. Thus $\parallel\widehat{\nu}\parallel_q \ll \parallel\widehat{\mu}\parallel_q$ and so (ii) is proved.

To prove (iii) for $\epsilon>0$, let $S_n = \{|f| > \frac{1}{n\|\widehat{\mu}\|_q(S)}\}$ and denote $F_n = \cup_{k=n}^{\infty} S_k$. Then $\{F_n\}$ is a decreasing sequence of sets with $F_n\searrow\emptyset$. For every $q\in Q$,

$$\begin{aligned}\sup_{y'\leq q}\left|y'\int_{S\cap S_n{}^c} f\,d\mu\right| &= \sup_{y'\leq q}\int_{S\cap S_n{}^c}|f|d|y'\mu|\\ &\leq \frac{1}{n\|\widehat{\mu}\|_q(S)}\sup_{y'\leq q}\int_{S\cap S_n{}^c} d|y'\mu|\\ &= \frac{1}{n\parallel\widehat{\mu}\parallel_q(S)}\parallel\widehat{\mu}\parallel_q(S\cap S_n{}^c)\leq\frac{1}{n}\end{aligned}$$

whence $\parallel\widehat{\mu}\parallel_q(S\cap S_n{}^c) = \sup_{y'\leq q}|y'\mu|_q(S\cap S_n{}^c) < \frac{1}{n}$. Moreover, as $S\cap F_n{}^c = \{s\in S : |f|=0\}$, for each $y'\in Y'$ we have $\int_{S\cap F_n{}^c}|f|d|y'\mu| = 0$, and therefore for every $q\in Q$, $\sup_{y'\leq q}|y'\widehat{\mu}|_q(S\cap F_n{}^c)=0$, that is $\parallel\widehat{\nu}\parallel_q(S\cap F_n{}^c)=0$.

THEOREM 3.5. *Let (f_n) be a sequence of μ -integrable functions such that*

(i) *$f_n \to f$ μ-a.e. on S in $||\widehat{\mu}||_q$-measure,*
(ii) *$\int_E f_n d\mu \ll \mu$ (in the sense of the $||\widehat{\mu}||_q$-semivariation) uniformly with respect to n.*

Then f is μ-integrable whenever Y is sequentially complete and

$$\lim_{n\to\infty}\int_E f_n d\mu = \int_E f d\mu \text{ uniformly for } E \in B(S).$$

PROOF. Since the sequence (f_n) is $y'\mu$-integrable, by applying the dominated convergence theorem, we see the following statement

$$\int_E f dy'\mu = \lim_{n\to\infty}\int_E f_n dy'\mu \text{ for } E \in B(S).$$

For $\epsilon > 0, n, k \in N$ fixed, let $E_{n,k} = \{|f_n - f_k| > \epsilon\}$ and $E \in B(S)$. Then $E_{n,k}$ is a decreasing sequence of sets with $E_{n,k} \searrow \emptyset$. Now for every $q \in Q$,

$$\begin{aligned} q[\int_E (f_n - f_k)d\mu] &\le q(\int_{E\cap E_{n,k}^c}(f_n - f_k)d\mu) + q(\int_{E\cap E_{n,k}}(f_n - f_k)d\mu) \\ &\le \sup_{y'\le q}\int_{E\cap E_{n,k}^c}|f_n - f_k|d|y'\mu| + \sup_{y'\le q}\int_{E\cap E_{n,k}}|f_n - f_k|d|y'\mu| \\ &\le \sup_{y'\le q}\int_{E\cap E_{n,k}^c}|f_n - f_k|d|y'\mu| + \sup_{y'\le q}\int_{E\cap E_{n,k}}|f_n|d|y'\mu| \\ &\quad + \sup_{y'\le q}\int_{E\cap E_{n,k}}|f_k|d|y'\mu|. \end{aligned}$$

From (ii) for $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $||\widehat{\mu}||_q(E) < \delta$, then $\sup_{y'\le q}\int_E |f_n|d|y'\mu| < \epsilon$ for all n. As for n, k suitably large $||\widehat{\mu}||_q(E \cap E_{n,k}) < \delta(\frac{\epsilon}{2})$, then $\sup_{y'\le q}\int_{E\cap E_{n,k}}|f_n|d|y'\mu| < \frac{\epsilon}{2}$ for every $n \in N$. Since $E_{n,k} = \{|f_n - f_k| > \epsilon\}$, $\| \widehat{\mu} \|_q (E_{n,k}) \to 0$ and for every $q \in Q$,

$$\sup_{y'\leq q}\int_{E\cap E^c_{n,k}} |f_n - f_k| d|y'\mu|$$
$$\leq \epsilon \sup_{y'\leq q} |y'\mu|(E\cap E^c_{n,k}) = \epsilon\|\widehat{\mu}\|_q(E\cap E^c_{n,k}) \leq \epsilon\|\widehat{\mu}\|_q(S).$$

Thus for every $q \in Q$, $q(\int_E (f_n - f_k)d\mu) < \epsilon[1+ \| \widehat{\mu} \|_q (S)]$ for all n, k and $E \in B(S)$. So it follows that $(\int_E f_n d\mu)$ is Cauchy in Y uniformly with respect to $q \in Q$. Thus there exists $y_E = \lim_{n\to\infty}\int_E f_n d\mu$. Since Y is sequentially complete, there exists an element y_E in Y such that

$$y'(y_E) = \lim_{n\to\infty}\int_E f_n d(y'\mu) = \int_E f d(y'\mu) \text{ for each } E\in B(S).$$

Therefore f is μ-integrable and $\int_E f d\mu = \lim_{n\to\infty}\int_E f_n d\mu$.

THEOREM 3.6. *If Y is sequentially complete and for $\epsilon > 0$ and $E_n \in B(S)$ for all n, $\| \widehat{\mu} \|_q (\cdot)$ is continuous at ϕ on $B(S)$, then every bounded measurable function $f : S \to X$ is μ-integrable.*

PROOF. Since f is a bounded measurable function, there is a sequence (f_n) of simple functions such that converges to f μ-a.e. on S and $\| f_n \|_S \leq \| f \|_S$ for all n. Let $\epsilon > 0$ be fixed, $E_n = \cup_{k=n}^{\infty}\{s \in S : |f - f_k| > \epsilon\}$. Then for each $y' \in Y'$ there exists a positive integer N such that $\| \widehat{\mu} \|_q (E_n) < \epsilon$ for all $n \geq N$. So

$$\int_E |f - f_k| d|y'\mu| \leq \int_{E\cap E^c_n} |f - f_k| d|y'\mu| + \int_{E\cap E_n} |f - f_k| d|y'\mu|$$
$$\leq \epsilon\|\widehat{\mu}\|_q(E\cap E^c_n) + 2M\|\widehat{\mu}\|_q(E\cap E_n)$$
$$< \epsilon(\|\widehat{\mu}\|_q(E\cap E^c_n) + 2M), \text{ where } M = \| f \|_S .$$

Thus f is $y'\mu$-integrable and $\int_E f dy'\mu = \lim_{n\to\infty}\int_E f_n dy'\mu$ for each $y' \in Y$. By the assumption for each $\epsilon > 0$, there exists an integer $N(> 0)$ such that $\| \widehat{\mu} \|_q (E_n) < \epsilon$ for $n \geq N$ and therefore

$$q(\int_E f_n d\mu - \int_E f_k d\mu) \leq \epsilon(\| \widehat{\mu} \|_q (E\cap E^c_n) + 2M)$$
$$+ \epsilon(\| \widehat{\mu} \|_q (E\cap E^c_k) + 2M) \text{ for all } n, k \geq N,\ E \in B(S).$$

Since Y is sequentially complete, there exists an element $y_E \in Y$, $y_E = \int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$. So f is μ-integrable.

DEFINITION 3.7[5]. A measureable function $f : S \longrightarrow X$ is said to be μ-integrable if there exists an X-valued sequence (f_n) of simple functions such that (i) $f_n \longrightarrow f$ μ-a.e. (ii) given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\| \widehat{\mu} \|_q (E) < \delta$, $E \in B(S)$ implies $q(\int_E f_n d\mu) < \epsilon$ for all $n = 1, 2, \cdots$.

THEOREM 3.8. *Let (f_n) be a sequence of μ-integrable functions which converges to f μ-a.e. and let $g : S \rightarrow X$ be a μ-integrable function such that $|f_n| \leq |g|$ for $n = 1, 2, \ldots$. Then f is μ-integrable whenever Y is sequentially complete.*

PROOF. Let $E_n = \cup_{n=1}^{\infty}\{|f_n - f| > \epsilon|g|\}$. Then $\{E_n\}$ is decreasing sequence of sets with $E_n \searrow \emptyset$. So, given $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that $\| \widehat{\mu} \|_q (E_n) < \frac{\epsilon}{8|g|}$ and since $q(f_n(s) - f(s)) \rightarrow 0$ uniformly on $S \cap E_n^c$,

$$q(f_n(s) - f(s)) < \frac{\epsilon}{4(1+ \| \widehat{\mu} \|_q (S))} \text{ for all } n \geq N,\ s \in S \cap E_n^c.$$

Also, since $|f_n| \leq |g|$ and $|f| \leq |g|$ μ-a.e., there exists $F \in B(S)$ such that $S \cap F^c$ is μ-null and $|f_n - f| \leq 2|g|$ for all $n \geq N$, $s \in F$. Thus

$$\begin{aligned}
\int_E |f - f_n| d|y'\mu| &\leq \int_{E\cap E_n^c} |f - f_n| d|y'\mu| + \int_{E\cap E_n} |f - f_n| d|y'\mu| \\
&\leq \frac{\epsilon}{4(1+ \| \widehat{\mu} \|_q (S))} \| \widehat{\mu} \|_q (E \cap E_n^c) \\
&+ \int_{(E\cap E_n)\cap F} |f_n - f| d|y'\mu| + \int_{(E\cap E_n)\cap F^c} |f_n - f| d|y'\mu| \\
&\leq \frac{\epsilon}{4} + 2 \| g \| \cdot \| \widehat{\mu} \|_q ((E \cap E_n) \cap F) \\
&\leq \frac{\epsilon}{4} + 2 \| g \| \cdot \| \widehat{\mu} \|_q (E_n) = \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Now let k, $n \geq N$ and $E \in B(S)$. We have $q(\int_E (f_n - f_k)) < \epsilon$. So $(\int_E f_n d\mu)$ is Cauchy uniformly for each $E \in B(S)$. Since Y is sequentially complete, there is an element $y_E = \int_E f d\mu$, which means $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$.

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