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REGULARLY DISSIPATIVE OPERATOR AND INTERPOLATION SPACES

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1. Introduction

Let H and V be complex Hilbert spaces such that the embedding $V \subset H$ is continuous. The inner product and norm in H are denoted by $(\cdot, \cdot)$ and $|\cdot|$ The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. Hence we may regard that

$$(1.1) \qquad ||u||_* \leq |u| \leq ||u||, \quad u \in V.$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$(1.2) \qquad \text{Re } a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A be the operator associated with the sesquilinear form $-a(\cdot, \cdot)$

$$(1.3) \qquad (Au, v) = -a(u,v), \quad u, \ v \in V.$$

The operator A defined by (1.3), using a bounded sesquilinear form satisfying (1.2), is called a regularly dissipative operator

In this paper we will show that A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^* and A is positive definite and self adjoint if $a(u,v)$ is symmetric. Finally, we will deal with interpolation spaces between the initial Hilbert space H and the domain of the regularly dissipative operator A by the J- and K-methods as in Butzer and Berens [1] and [2](see also [4,5]). In [3,4], interpolation spaces generated by C_0-semigroup and analytic semigroup operators were established

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2. Preliminaries

Let X and Y be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}$ such that the embedding mapping of both X and Y in $\mathcal{X}$ is continuous. Let $X \cap Y$ be a dense subspace in both X and Y. For $1 < p < \infty$, we denote by $L_*^p(X)$ the Banach space of all functions $t \to u(t)$, $t \in (0, \infty)$ and $u(t) \in X$, for which the mapping $t \to u(t)$ is strongly measurable with respect to the measure dt/t and the norm $||u||_{L_*^p(X)}$ is finite, where

$$||u||_{L_*^p(X)} = \{\int_0^\infty ||u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

For $0 < \theta < 1$, set

$$||t^\theta u||_{L_*^p(X)} = \{\int_0^\infty ||t^\theta u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}},$$
$$||t^\theta u'||_{L_*^p(Y)} = \{\int_0^\infty ||t^\theta u'(t)||_Y^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

We now introduce a Banach space

$$V = \{u : ||t^\theta u||_{L_*^p(X)} < \infty, \quad ||t^\theta u'||_{L_*^p(Y)} < \infty\}$$

with norm

$$||u||_V = ||t^\theta u||_{L_*^p(X)} + ||t^\theta u'||_{L_*^p(Y)}.$$

DEFINITION 2.1. We define $(X, Y)_{\theta, p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, to be the space of all elements $u(0)$ where $u \in V$, that is,

$$(X, Y)_{\theta,p} = \{u(0) : u \in V\}.$$

For $0 < \theta < 1$ and $1 \leq p \leq \infty$, the space $(X, Y)_{\theta,p}$ is a Banach space with the norm

$$||a||_{\theta,p} = \inf\{||u|| : u \in V, \quad u(0) = a\}.$$

Furthermore, there is a constant $C_\theta > 0$ such that

$$||a||_{\theta,p} = C_\theta \inf\{||t^\theta u||^{1-\theta}_{L^p_*(X)}||t^\theta u'||^\theta_{L^p_*(Y)} : u(0) = a, \quad u \in V\}$$

as is seen in [2]. It is known that $(X,X)_{\theta,p} = X$ for $0 < \theta < 1$ and $1 \le p \le \infty$ and

$$(X,Y)_{\theta,p} \subset (X,Y)_{\theta',p}, \quad 0 < \theta < \theta' < 1$$

where $X \subset Y$ satisfying that there exists a constant $c > 0$ such that $||u||_Y \le c||u||_X$.

Let X be a Banach space with norm $||\cdot||$ and $T(t)$ be a C_0-semigroup with infinitesimal generator A. Then its domain $D(A)$ is a Banach space with the graph norm $||x||_{D(A)} = ||Ax|| + ||x||$. The following result is obtained from Theorem 3.1 in [3]

PROPOSITION 2.1. *Let A be the generator of a C_0-semigroup $T(t)$. Then for $0 < \theta < 1$, $1 < p < \infty$, we have that*

$$(D(A),X)_{\theta,p} = \{x \in X : \int_0^\infty (t^{\theta-1}||T(t)x - x||)^p \frac{dt}{t} < \infty\}.$$

Let $T(t)$ be an analytic semigroup with infinitesimal generator A. We may assume that

$$||T(t)|| \le M, \quad ||AT(t)|| \le \frac{K}{t}$$

for some positive constants M, K and $t \ge 0$ The following result is obtained from Theorem 3.1 in [4].

PROPOSITION 2.2. *Let A be the generator of an analytic semigroup $T(t)$. Then for $0 < \theta < 1$, $0 \le t$, we have*

$$(D(A),X)_{\theta,p} = \{x \in X : \int_0^\infty (t^\theta||AT(t)x||)^p \frac{dt}{t} < \infty\}.$$

3. Regularly dissipative operators and interpolation spaces

Let $a(\cdot,\cdot)$ be a sesquilinear form defined in $V \times V$, that is, for each $u, v \in V$ there corresponds a complex number $a(u,v)$ which is linear in u and antilinear in v:

$$a(u_1 + u_2, v) = a(u_1, v) + a(u_2, v),$$
$$a(u, v_1 + v_2) = a(u, v_1) + a(u, v_2),$$
$$a(\lambda u, v) = \lambda a(u, v), \quad a(u, \lambda v) = \overline{\lambda} a(u, v).$$

We assume that $a(u,v)$ is bounded, i.e., there exists a constant M such that

$$|a(u,v)| \leq M||u||||v||, \quad u, v \in V \tag{3.1}$$

and satisfies Gårding's inequality

$$\text{Re } a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0. \tag{3.2}$$

Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u,v), \quad u,\ v \in V.$$

When $a(u,v)$ with $u \in V$ fixed is considered as a functional of v, it is an element of V^* . Therefore, using an element $f \in V^*$, we can write $Au = f$ in the sense of the following Lax-Milgram theorem.

LEMMA 3 1 *Let X be a Hilbert space, whose inner product and norm will be denoted by $(\cdot,\cdot)$ and $||\cdot||$, respectively. Assume that $b(u,v)$ is sesquilnear form defined on $X \times X$ and that there exists positive constants C and c such that*

$$|b(u,v)| \leq C||u||||v||, \tag{3.3}$$
$$|b(u,u)| \geq c||u||^2 \tag{3.4}$$

for every $u, v \in X$. Under these conditions, if $f \in X^*$, then there uniquely exists an element $u \in X$ such that $f(v) = b(u,v)$ for every $v \in X$.

The realization for the operator A in H, which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}, \tag{3.5}$$

is denoted by A_H.

In what follows we assume that (3.2) holds for $c_1 = 0$:

$$\text{Re } a(u,u) \geq c_0 ||u||^2, \quad c_0 > 0. \tag{3.6}$$

Let $f \in V^*$, from (3 1), (3.6) and Lemma 3.1 it follows that there exists a $v \in V$ such that $(f, v) = a(u, v)$ for all $v \in V$, i.e., $f = Au$, and hence $R(A) = V^*$. Combining this result with (3 5), we have $R(A_H) = H$ From (3.1) and (3 6), we have that

$$c_0||u|| \leq ||Au||_* \leq M||u||. \tag{3.7}$$

Thus A is an isomorphism from V onto V^*.

LEMMA 3.2. *For $Re\lambda \leq 0$ there exists a bounded inverse of $A - \lambda$ which has various bounds for every f of H or V^*:*

$$|(A-\lambda)^{-1}f| \leq M_1|\lambda|^{-1}|f|, \tag{i}$$

$$||(A-\lambda)^{-1}f||_* \leq M_1|\lambda|^{-1}||f||_*, \tag{ii}$$

where $M_1 = 1 + M/c_0$. Here, M and c_0 are the constants in (3.7) and (3.6), respectively.

PROOF. For $u, v \in V$, put $b(u,v) = ((A-\lambda)u, v)$ with $\text{Re}\,\lambda \leq 0$, then

$$b(u,v) = a(u,v) - \lambda(u,v), \quad v \in V.$$

From (3.1) and (3 2) it follows

$$\begin{aligned} |b(u,v)| &\leq |a(u,v)| + |\lambda||(u,v)| \\ &\leq M||u||||v|| + |\lambda||u||v| \\ &\leq (M + |\lambda|)||u||||v|| \end{aligned}$$

and

$$|b(u,u)| \geq \operatorname{Re} b(u,u) = \operatorname{Re} a(u,u) - \operatorname{Re}\lambda|u|^2 \geq c_0||u||^2.$$

Hence, by Lemma 3.1 for every $f \in V^*$ there uniquely exists $u \in V$ such that

$$(f,v) = b(u,v) = ((A-\lambda)u, v),$$

i.e., $f = (A-\lambda)u$. Since

$$(f,v) = a(u,v) - \lambda(u,v)$$

we have

$$(f,u) = a(u,u) - \lambda|u|^2, \tag{3.8}$$

$$\operatorname{Re}(f,u) = \operatorname{Re} a(u,u) - \operatorname{Re}\lambda|u|^2. \tag{3.9}$$

From (3.9) it follows

$$c_0||u||^2 \leq |f||u|,$$

and hence,

$$\begin{aligned} |\lambda||u|^2 &= |a(u,u) - (f,u)| \leq M||u||^2 + |f||u| \\ &\leq \frac{M}{c_0}|f||u| + |f||u| = (\frac{M}{c_0}+1)|f||u|. \end{aligned}$$

It implies that $|u| \leq M_1|\lambda|^{-1}|f|$ where $M_1 = M/c_0 + 1$, which proves (i).

From (3.9) we have

$$c_0||u||^2 - \operatorname{Re}\lambda|u|^2 \leq ||f||_*||u||,$$

thus

$$||u|| \leq \frac{1}{c_0}||f||_*, \tag{3.10}$$

from which it follows that

$$\begin{aligned}|\lambda||(u,v)| &\le M||u||||v|| + ||f||_*||v|| \\ &\le Mc_0^{-1}||f||_*||v|| + ||f||_*||v|| \\ &\le (\frac{M}{c_0}+1)||f||_*||v|| = M_1||f||_*||v||\end{aligned}$$

for every $v \in V$. It implies

$$\frac{|(u,v)|}{||v||} \le \frac{M_1}{|\lambda|}||f||_*, \quad v \in V$$

i.e.,

$$||u||_* \le \frac{M_1}{|\lambda|}||f||_*, \quad v \in V.$$

The proof of (ii) is complete.

THEOREM 3.1. *$-A$ and $-A_H$ generate analytic semigroups in V^* and H, respectively.*

PROOF. The half plane $\{\lambda . \mathrm{Re}\lambda > 0\}$ is contained in $\rho(-A)$ from Lemma 3.2 where $\rho(-A)$ stands for the resolvent set of $-A$. Let λ_0 be a complex number such that $\mathrm{Re}\,\lambda_0 = 0$ and $\mathrm{Im}\lambda_0 > 0$. Then if $|\lambda - \lambda_0| < |\lambda_0|/M_1$ where M_1 is the constant in Lemma 3.2, from (ii) of Lemma 3.2 we have

$$||(\lambda-\lambda_0)(\lambda_0 - A)^{-1}||_* \le |\lambda - \lambda_0|\frac{M_1}{|\lambda_0|}.$$

Note

$$\begin{aligned}\lambda - A &= \lambda - \lambda_0 + \lambda_0 - A \\ &= \{I + (\lambda-\lambda_0)(\lambda_0 - A)^{-1}\}(\lambda_0 - A)\end{aligned}$$

and

$$(\lambda - A)^{-1} = (\lambda_0 - A)^{-1}\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(\lambda_0 - A)^{-n}.$$

Hence

$$\begin{aligned}\|(\lambda - A)^{-1}\|_* &\le \|(\lambda_0 - A)^{-1}\|_* \sum_{n=0}^{\infty} |\lambda - \lambda_0|^n \|(\lambda_0 - A)^{-1}\|_*^n \\ &\le \frac{M_1}{|\lambda_0|} \sum_{n=0}^{\infty} |\lambda - \lambda_0|^n (\frac{M_1}{|\lambda_0|})^n = \frac{M_1}{|\lambda_0|} \frac{1}{1 - \frac{M_1|\lambda - \lambda_0|}{|\lambda_0|}} \\ &= \frac{M_1}{|\lambda_0| - M_1|\lambda - \lambda_0|}.\end{aligned}$$

It is a necessary and sufficient condition for A to generate an analytic semigroup in V^*(see Theorem 3.3.1 or Remark 3.3.2 of [7]). Since $A_H u = Au$ for all $u \in D(A_H)$, from (i) of Lemma 3.2 we immediately obtain that A_H generates an analytic semigroup in H.

DEFINITION 3.1. Let A be a linear operator in a Hilbert space H and its domain is assumed to be dense. The operator A is called a dissipative operator if $\text{Re}\,(Au, u) \le 0$ for all $u \in D(A)$. If $\text{Re}\,(Au, u) \ge 0$ for all $u \in D(A)$, that is, $-A$ is a dissipative operator, A is said to be an accretive operator. A dissipative operator which extends a dissipative operator A is called a dissipative extension of A. An operator A is said to be maximal dissipative if its any dissipative extension is A itself. Accretive extensions and maximal accretive operators are defined similarly.

A sesquilinear form $a^*(u, v)$ defined by $a^*(u, v) = \overline{a(v, u)}$ is called an *adjoint sesquilinear form* If $a(u, v)$ satisfied (3.1), (3.2) or (3.6), so, correspondingly, does $a^*(u, v)$. Let A' and A'_H be operators defined by $a^*(u, v)$ in ways similar to A and A_H, respectively:

Let $u \in V$. If there exists an $f \in V^*$ such that $a^*(u, v) = (f, v)$ all $v \in V$, then $u \in D(A')$ and $A'u = f$. $a^*(u, v) = (A'u, v)$ for all $u, v \in V$.

Assume again that (3.6) is satisfied. Then, as in the cases of A_H and A, we have $R(A'_H) = X$ and $R(A') = V^*$ for A'_H and A'.

LEMMA 3.3. *$D(A_H)$ is dense in V. Therefore, it is also dense in H.*

PROOF. It is enough to show that $f \in V^*$ and if $(f, v) = 0$ for all $v \in D(A_H)$, then $f = 0$. Since $R(A') = V^*$, there exists a $u \in V$ such that $f = A'u$. If $v \in D(A_H)$, we have

$$(A_H v, u) = a(v, u) = \overline{a^*(u, v)} = \overline{(A'u, v)} = \overline{(f, v)} = 0$$

This, together with $R(A_H) = H$, implies $u = 0$. Hence, $f = 0$.

Since $\mathrm{Re}\,(A_H u, u) = \mathrm{Re}\,a(u, u) \geq c_0\|u\|^2 \geq 0$ for any $u \in D(A_H)$, the operator A_H is accretive.

DEFINITION 3.2. The operator A_H associated with a sesquilinear form satisfying (3.1) and (3.6), is called a regularly accretive operator. If $-A_H$ is regularly accretive, A_H is called a regularly dissipative operator.

THEOREM 3.2. *Let A_H^* be an adjoint of A_H when the latter is viewed as an operator in H. Then $A'_H = A_H^*$.*

PROOF. Let $u \in D(A_H)$ and $v \in D(A'_H)$. Then we find

$$(A_H u, v) = a(u, v) = \overline{a^*(v, u)} = \overline{(A'_H v, u)} = (u, A'_H v).$$

This shows $A'_H \subset A_H^*$. Let $u \in D(A_H^*)$ and put $A_H^* u = f$. Since $R(A'_H) = H$, there exists a $w \in D(A'_H)$ such that $f = A'_H w$. The relation $A'_H \subset A_H^*$ implies $f = A_H^* w$. Since $0 \in \rho(A_H)$ we have $0 \in \rho(A_H^*)$. Therefore, $u = w \in D(A'_H)$ and, hence, $A'_H = A_H^*$.

From now on, both A_H and A are denoted simply by A. We also denote A' by A^*. Therefore, for any $u, v \in V$, we have

$$a(u, v) = (Au, v), \qquad a^*(u, v) = (A^*u, v).$$

This notation will not cause any confusion.

When $a^*(u, v) = a(u, v)$ holds for all $u, v \in V$, the sesquilinear form $a(u, v)$ is said to be symmetric. In this case, by Theorem 3.2, an operator A in X is self-adjoint. It is evident that $a(u, u)$ is a real number for each $u \in V$. Since, by (3.6), we have

$$(Au, u) = a(u, u) \geq c_0|u|^2$$

for all $u \in D(A)$, the operator A is bounded from below. In particular, A is positive definite if (3.6) is satisfied.

THEOREM 3.3. *If $a(u,v)$ is a symmetric sesquilinear form satisfying (3.1) and (3.6), then A is positive definite and self-adjoint, $D(A^{1/2}) = V$ and*

$$a(u,v) = (A^{1/2}u, A^{1/2}v), \quad u, v \in V. \tag{3.11}$$

PROOF. According to this assumption, for each $u \in D(A)$ we have

$$c_0\|u\|^2 \le a(u,v) = (Au,u) = |A^{1/2}u|^2. \tag{3.12}$$

Let u be an arbitrary element of $D(A^{1/2})$. For each natural number n we put $u_n = (1+n^{-1}A)^{-1}u$. Then $u_n \in D(A)$ and we can show by the using the spectral resolution that

$$u_n \to u$$

and

$$A^{1/2}u_n = (1+n^{-1}A)^{-1}A^{1/2}u \to A^{1/2}u$$

in H as $n \to \infty$. By applying (3.12) to $u_n - u_m$, it is found that $\{u_n\}$ is a Cauchy sequence in V. Since $u_n \to u$ in H, so it does in V ; hence, $D(A^{1/2}) \subset V$. Applying (3.12) to u_n and let $n \to \infty$, then we obtain $c_0\|u\|^2 \le |A^{1/2}u|^2$. On the other hand, if we let $u \in V$, by Lemma 3.3 there exists a sequence $\{u_j\}$ of elements of $D(A)$ such that $\|u_j - u\| \to 0$. Since

$$|A^{1/2}(u_j - u_k)|^2 = a(u_j - u_k, u_j - u_k) \le M\|u_j - u_k\|^2,$$

$\{A^{1/2}u_j\}$ is a Cauchy sequence in H. Since $A^{1/2}$ is a closed operator, $u \in D(A^{1/2})$ and thus we have obtained $D(A^{1/2}) = V$. Equation (3.11) can be easily verified

REMARK. For $c_1 > 0$, replace $a(u,v)$ by $a(u,v) + c_1(u,v)$ and A by $A + c_1$, respectively, then the conclusion of Theorem 3.3 still holds.

Let H_1 and H_2 be Hilbert spaces with inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$, respectively. Their norms will be denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Assume that H_1 is dense subspace in H_2 and the injection of H_1 into H_2 is continuous.

Now we can apply the previous Theorem to the case with H_1, H_2 and $(\cdot,\cdot)_1$ in place of V, H and $a(\cdot,\cdot)$,respectively. Therefore, if we put

$$(Au, v)_2 = (u, v)_1$$

then A is positive definite and self adjoint on H_2, $D(A^{1/2}) = H_1$ and

$$(u, v)_1 = (A^{1/2}u, A^{1/2}v)_2, \quad u, v \in H_1.$$

THEOREM 3.4. *Put* $\Lambda = A^{1/2}$ *Then* Λ *is positive definite and self adjoint on* H_2, $D(\Lambda) = H_1$, *and*

$$(u, v)_1 = (\Lambda u, \Lambda v)_2, \quad u, v \in H_1.$$

Furthermore,

$$(H_1, H_2)_{\theta,2} = D(\Lambda^{1-\theta}), \quad 0 < \theta < 1. \tag{3.13}$$

PROOF. We know that $-\Lambda$ generates analytic semigroup and if

$$\Lambda = \int_0^\infty \lambda dE(\lambda)$$

is the spectral resolution of the self adjoint operator Λ then

$$e^{-t\Lambda} = \int_0^\infty e^{-t\lambda} dE(\lambda).$$

By Proposition 2 2, it holds that

$$\begin{aligned}(H_1, H_2)_{\theta,2} &= (D(\Lambda), H_2)_{\theta,2} \\ &= \{x \in H_2 : \int_0^\infty (t^\theta ||\Lambda e^{-t\Lambda}x||_2)^2 \frac{dt}{t}\} < \infty.\end{aligned}$$

The proof of (3.13) is a consequence of following inequality

$$\int_0^\infty (t^\theta ||\Lambda e^{-t\Lambda}x||)^2 \frac{dt}{t} = \int_0^\infty t^{2\theta-1}||\Lambda e^{-t\Lambda}x||_2^2 dt$$
$$= \int_0^\infty t^{2\theta-1}|| \int_0^\infty \lambda e^{-t\lambda} dE(\lambda)x||_2^2 dt$$
$$= \int_0^\infty t^{2\theta-1} \int_0^\infty \lambda^2 e^{-2t\lambda} d||E(\lambda)x||_2^2 dt$$
$$= \int_0^\infty \lambda^2 \int_0^\infty t^{2\theta-1} e^{-2t\lambda} dt d||E(\lambda)x||_2^2$$
$$= \int_0^\infty \lambda^2 \int_0^\infty (\frac{t}{2\lambda})^{2\theta-1} e^{-t} \frac{dt}{2\lambda} d||E(\lambda)x||_2^2$$
$$= \int_0^\infty \lambda^2 (2\lambda)^{-2\theta} \int_0^\infty t^{2\theta-1} e^{-t} dt d||E(\lambda)x||_2^2$$
$$= 2^{-2\theta}\Gamma(2\theta) \int_0^\infty \lambda^{2-2\theta} d||E(\lambda)||_2^2$$
$$= 2^{-2\theta}\Gamma(2\theta)||\Lambda^{1-\theta}x||_2^2,$$

where the Γ is the Gamma function.

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