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COMMON FIXED POINTS OF Φ-CONTRACTIVE MAPPINGS

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ABSTRACT In this paper, we give some common fixed point theorems for compatible mappings in metric spaces, and also give an example to illustrate our main theorems Our results extend the results of S M Kang, Y. J Cho and G Jungck [9].

1. Introduction

The most well-known fixed point theorem proved by S. Banach in 1922 is so called the Banach Contraction Principle, which asserts that any contractive mapping from a complete metric space into itself has a unique fixed point in a complete metric space. By using the more generalized contractive condition, G. E. Hardy and T. D Rogers [4] extended the Banach Contraction Principle.

In 1976, G. Jungck [5] initially proved a common fixed point theorem for commuting mappings which generalizes the Banach Contraction Principle

In 1982, S. Sessa [13] introduced a generalization of commuting mappings, which is called weakly commuting mappings, and proved some common fixed point theorems for these mappings which generalize the

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results of K. M. Das and K. V. Naik [1]. Further, G. Jungck [6] introduced the concept of more generalized commuting mappings, so called compatible mappings, which is more general than that of weakly commuting mappings. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of S. Park and J. S. Bae [12]. By employing compatible mappings instead of commuting mappings and using four mappings instead of three mappings, G. Jungck [7] extended the results of M. S. Khan and M. Imdad [10], S. L. Singh and S. P. Singh [14] and also obtained an interesting result in his consecutive paper [8].

Also, by using compatible mappings, S. M. Kang, Y. J. Cho and G. Jungck [9] generalized the results of X. P. Ding [2], M. L. Diviccaro and S. Sessa [3] and G. Jungck [7] in metric spaces.

In this paper, we give some common fixed point theorems for compatible mappings in metric spaces, and also give an example to illustrate our main theorems. Our results extend the results of S. M. Kang, Y. J. Cho and G. Jungck [9].

2. Preliminaries

For some definitions and properties in this paper, we refer to G. Jungck [6], [7].

DEFINITION 2.1. Let A and B be mappings from a metric space (X, d) into itself. Then the mappings A and B are said to be *compatible* if

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t in X.

Now, we give some properties of compatible mappings in a metric space for our main theorems:

PROPOSITION 2.1. *Let A and B be compatible mappings from a metric space (X, d) into itself. If $At = Bt$ for some t in X, then $ABt = BBt = BAt = AAt$.*

PROPOSITION 2.2. *Let A and B be compatible mappings from a metric space (X,d) into itself. If $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t in X, then $\lim_{n\to\infty} BAx_n = At$ if A is continuous at t.*

3. Fixed Point Theorems in Metric Spaces

Throughout this paper, let N and R^+ be the sets of all natural numbers and non-negative real numbers, respectively.

Let p is a positive integer. Assume that $\phi : (R^+)^5 \to R^+$ be a function. We say that ϕ satisfies the *condition* (Φ) if

(i) ϕ is upper-semicontinuous and non-decreasing in each coordinate variables,

(ii) for each $t > 0$,

$$\varphi(t) = \max\{\phi(0,0,t,t,t),\ \phi(t,t,t,2^p t,0),\ \phi(t,t,t,0,2^p t)\} < t,$$

where $\varphi : R^+ \to R^+$ is a real-valued function.

The above the condition (Φ) is considered by X. P. Ding [2]. We denote $d^p(x,y) = [d(x,y)]^p$.

Let A, B, S and T be mappings from a metric space (X,d) into itself such that

$$A(X) \subset T(X), \quad B(X) \subset S(X), \tag{3.1}$$

$$d^p(Ax,By) \le \phi(d^p(Ax,Sx), d^p(By,Ty), d^p(Sx,Ty), d^p(Ax,Ty), d^p(By,Sx)) \tag{3.2}$$

for all x, y in X and $p \in N$, where ϕ satisfies the condition (Φ). Then, for any arbitrary point x_0 in X, by $A(X) \subset T(X)$, we can choose a point x_1 in X such that $y_1 = Tx_1 = Ax_0$ and, for this point x_1, by $B(X) \subset S(X)$, we can choose a point x_2 in X such that $y_2 = Sx_2 = Bx_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that $y_1 = Tx_1 = Ax_0$,

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \tag{3.3}$$

for every $n \in N$.

For our main theorems, we need the following lemmas:

LEMMA 3.1. [11] *Suppose that $\varphi : R^+ \to R^+$ is upper-semicontinuous and non-decreasing. Then, for any $t > 0$, $\varphi(t) < t$ if and only if $\lim_{n\to\infty} \varphi^n(t) = 0$, where φ^n denotes the n-times composition of φ.*

LEMMA 3.2. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (3.1) and (3.2). Then*

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0,$$

where $\{y_n\}$ is the sequence in X defined by (3.3).

PROOF. By (3.2) and (3.3), we have

$$\begin{aligned}
&d^p(y_{2n+1}, y_{2n+2}) = d^p(Ax_{2n}, Bx_{2n+1}) \\
&\leq \phi(d^p(Ax_{2n}, Sx_{2n}), d^p(Bx_{2n+1}, Tx_{2n+1}), \\
&\qquad d^p(Sx_{2n}, Tx_{2n+1}), d^p(Ax_{2n}, Tx_{2n+1}), d^p(Bx_{2n+1}, Sx_{2n})) \\
&= \phi(d^p(y_{2n+1}, y_{2n}), d^p(y_{2n+2}, y_{2n+1}), \\
&\qquad d^p(y_{2n}, y_{2n+1}), d^p(y_{2n+1}, y_{2n+1}), d^p(y_{2n}, y_{2n+2})) \\
&\leq \phi(d^p(y_{2n+1}, y_{2n}), d^p(y_{2n+2}, y_{2n+1}), \\
&\qquad d^p(y_{2n}, y_{2n+1}), 0, [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]^p).
\end{aligned}$$

If $d(y_{2n+1}, y_{2n+2}) > d(y_{2n}, y_{2n+1})$ in the above inequality, then we have

$$\begin{aligned}
d^p(y_{2n+1}, y_{2n+2}) &\leq \phi(d^p(y_{2n+1}, y_{2n+2}), d^p(y_{2n+1}, y_{2n+2}), \\
&\qquad d^p(y_{2n+1}, y_{2n+2}), 0, 2^p d^p(y_{2n+1}, y_{2n+2})) \\
&\leq \varphi(d^p(y_{2n+1}, y_{2n+2})) \\
&< d^p(y_{2n+1}, y_{2n+2}),
\end{aligned}$$

which is a contradiction. Thus it follows that

$$\begin{aligned}
d^p(y_{2n+1}, y_{2n+2}) &\leq \phi(d^p(y_{2n}, y_{2n+1}), d^p(y_{2n}, y_{2n+1}), \\
&\qquad d^p(y_{2n}, y_{2n+1}), 0, 2^p d^p(y_{2n}, y_{2n+1})) \\
&\leq \varphi(d^p(y_{2n}, y_{2n+1})).
\end{aligned} \tag{3.4}$$

Similarly, we have

$$d^p(y_{2n+2}, y_{2n+3}) \leq \varphi(d^p(y_{2n+1}, y_{2n+2})). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$d^p(y_n, y_{n+1}) \leq \varphi(d^p(y_{n-1}, y_n)) \leq \cdots \leq \varphi^{n-1}(d^p(y_1, y_2)).$$

By Lemma 3.1, we obtain

$$\lim_{n\to\infty} d^p(y_n, y_{n+1}) = 0.$$

This completes the proof.

LEMMA 3.3. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Then the sequence $\{y_n\}$ in X defined by* (3.3) *is a Cauchy sequence in X.*

PROOF. In virtue of Lemma 3.2, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ such that, for each positive integer k, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \varepsilon. \tag{3.6}$$

For each positive integer k, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.6), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon, \quad d(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{3.7}$$

Then, for each even integer $2k$,

$$\begin{aligned}\varepsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

It follows from Lemma 3.2 and (3.7) that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon. \tag{3.8}$$

By the triangle inequality,

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)}),$$

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

From Lemma 3.2 and (3.8), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon. \tag{3.9}$$

By (3.2), we have

$$\begin{aligned} d^p(y_{2n(k)}, y_{2m(k)}) &\le [d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)})]^p \\ &= \lambda + d^p(y_{2n(k)+1}, y_{2m(k)}) \\ &= \lambda + d^p(Ax_{2n(k)}, Bx_{2m(k)-1}) \\ &\le \lambda + \phi(d^p(y_{2n(k)+1}, y_{2n(k)}), d^p(y_{2m(k)}, y_{2m(k)-1}), \\ &\qquad d^p(y_{2n(k)}, y_{2m(k)-1}), d^p(y_{2n(k)+1}, y_{2m(k)-1}), \\ &\qquad d^p(y_{2m(k)}, y_{2n(k)})), \end{aligned}$$

where

$$\begin{aligned} \lambda = {} & d^p(y_{2n(k)}, y_{2n(k)+1}) \\ & + \frac{p!}{(p-1)!} d^{p-1}(y_{2n(k)}, y_{2n(k)+1}) \cdot d(y_{2n(k)+1}, y_{2m(k)}) \\ & + \frac{p!}{2!(p-2)!} d^{p-2}(y_{2n(k)}, y_{2n(k)+1}) \cdot d^2(y_{2n(k)+1}, y_{2m(k)}) \\ & + \cdots \\ & + \frac{p!}{(p-1)!} d(y_{2n(k)}, y_{2n(k)+1}) \cdot d^{p-1}(y_{2n(k)+1}, y_{2m(k)}). \end{aligned}$$

Using Lemma 3.2, (3.8), (3.9), since ϕ satisfies the condition (Φ), we have

$$\varepsilon^p \le \phi(0, 0, \varepsilon^p, \varepsilon^p, \varepsilon^p) \le \varphi(\varepsilon^p) < \varepsilon^p,$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X. This completes the proof.

Now, we are ready to give our main theorems:

THEOREM 3.4. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Suppose that*
(3.10) *one of A, B, S and T is continuous,*
(3.11) *the two pairs A, S and B, T are compatible mappings.*
Then A, B, S and T have a unique common fixed point in X.

PROOF. Let $\{y_n\}$ be the sequence in X defined by (3.3). By Lemma 3.3, $\{y_n\}$ is a Cauchy sequence in X and hence it converges to some point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z.

Now, suppose that S is continuous. Since A and S are compatible mappings, it follows from Proposition 2.2 that

$$ASx_{2n}, \quad SSx_{2n} \to Sz \quad \text{as } n \to \infty.$$

By (3.2), we have

$$\begin{aligned} d^p(ASx_{2n}, Bx_{2n+1}) \leq \phi(&d^p(ASx_{2n}, SSx_{2n}), d^p(Bx_{2n+1}, Tx_{2n+1}), \\ &d^p(SSx_{2n}, Tx_{2n+1}), d^p(ASx_{2n}, Tx_{2n+1}), d^p(Bx_{2n+1}, SSx_{2n})). \end{aligned}$$

By letting $n \to \infty$ in the above inequality, we have

$$\begin{aligned} d^p(Sz, z) &\leq \phi(d^p(Sz, Sz), d^p(z, z), \\ &\qquad\qquad d^p(Sz, z), d^p(Sz, z), d^p(z, Sz)) \\ &= \phi(0, 0, d^p(Sz, z), d^p(Sz, z), d^p(z, Sz)) < d^p(Sz, z), \end{aligned}$$

which is a contradiction Thus we have $Sz = z$ Again, from (3.2), we obtain

$$\begin{aligned} &d^p(Az, Bx_{2n+1}) \\ &\leq \phi(d^p(Az, Sz), d^p(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d^p(Sz, Tx_{2n+1}), d^p(Az, Tx_{2n+1}), d^p(Bx_{2n+1}, Sz)). \end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned} d^p(Az, z) &\leq \phi(d^p(Az, Sz), d^p(z, z), \\ &\qquad\qquad d^p(Sz, z), d^p(Az, z), d^p(z, Sz)) \\ &= \phi(d^p(z, Az), 0, 0, d^p(Az, z), 0) < d^p(Az, z), \end{aligned}$$

which implies that $Az = z$. Since $A(X) \subset T(X)$, there exists a point $u \in X$ such that $z = Az = Tu$. Again, by (3.2), we have

$$\begin{aligned} d^p(z, Bu) &= d^p(Az, Bu) \\ &\le \phi(d^p(Az, Sz), d^p(Bu, Tu), \\ &\qquad d^p(Sz, Tu), d^p((Az, Tu), d^p(Bu, Sz)) \\ &= \phi(0, d^p(Bu, z), 0, 0, d^p(Bu, z)) \\ &< d^p(z, Bu), \end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible mappings and $Tu = Bu = z$, by Proposition 2.1, $TBu = BTu$ and hence $Tz = TBu = BTu = Bz$. Moreover, by (3.2), we have

$$\begin{aligned} d^p(Ax_{2n}, Bz) \le \phi(&d^p(Ax_{2n}, Sx_{2n}), d^p(Bz, Tz), \\ &d^p(Sx_{2n}, Tz), d^p(Ax_{2n}, Tz), d^p(Bz, Sx_{2n})). \end{aligned}$$

By letting $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} d^p(z, Bz) &\le \phi(0, 0, d^p(z, Bz), d^p(z, Bz), d^p(z, Bz)) \\ &< d^p(z, Bz), \end{aligned}$$

so that $z = Bz$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when T is continuous.

Now, suppose that A is continuous. Since A and S are compatible, it follows from Proposition 2.2 that

$$SAx_{2n}, \quad AAx_{2n} \to Az \quad \text{as } n \to \infty.$$

By (3.2), we have

$$\begin{aligned} &d^p(AAx_{2n}, Bx_{2n+1}) \\ &\le \phi(d^p(AAx_{2n}, SAx_{2n}), d^p(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d^p(SAx_{2n}, Tx_{2n+1}), d^p(AAx_{2n}, Tx_{2n+1})), d^p(Bx_{2n+1}, SAx_{2n}). \end{aligned}$$

By letting $n \to \infty$, we have

$$\begin{aligned} d^p(Az, z) &\leq \phi(0, 0, d^p(Az, z), d^p(Az, z), d^p(z, Az)) \\ &< d^p(Az, z), \end{aligned}$$

which is a contradiction. Thus we have $Az = z$. Hence $A(X) \subset T(X)$, there exists a point $u \in X$ such that $z = Az = Tu$. By using (3.2), we have

$$\begin{aligned} &d^p(AAx_{2n}, Bu) \\ &\leq \phi(d^p(AAx_{2n}, Sx_{2n}), d^p(Bu, Tu), \\ &\qquad d^p(SAx_{2n}, Tu), d^p(AAx_{2n}, Tu), d^p(Bu, SAx_{2n})). \end{aligned}$$

By letting $n \to \infty$, we obtain

$$\begin{aligned} d^p(z, Bu) &\leq \phi(0, d^p(Bu, z), d^p(z, Tu), d^p(z, Tu), d^p(Bu, z)) \\ &< d^p(z, Bu), \end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible mappings and $Tu = Bu = z$, by Proposition 2 1, $TBu = BTu$ and hence $Tz = TBu = BTu = Bz$. Moreover, by (3.2), we have

$$\begin{aligned} d^p(Ax_{2n}, Bz) \leq \phi(&d^p(Ax_{2n}, Sx_{2n}), d^p(Bz, Tz), \\ &d^p(Sx_{2n}, Tz), d^p(Ax_{2n}, Tz), d^p(Bz, Sx_{2n})). \end{aligned}$$

By letting $n \to \infty$, we obtain

$$d^p(z, Bz) \leq \phi(0, 0, d^p(z, Tz), d^p(Tz, z), d^p(Bz, z)) < d^p(z, Bz),$$

which means that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point $v \in X$ such that $z = Bz = Sv$. By using (3.2), we have

$$\begin{aligned} d^p(Av, z) = d^p(Av, Bz)& \\ &\leq \phi(d^p(Av, Sv), d^p(Bz, Tz), \\ &\qquad d^p(Sv, Tz), d^p(Av, Tz), d^p(Bz, Sv)) \\ &= \phi(d^p(Av, z), 0, 0, d^p(Av, z), 0) < d^p(Av, z), \end{aligned}$$

so that $Av = z$. Since A and S are compatible and $Sv = Av = z$, by Proposition 2.1, $SAv = ASv$ and so $Az = ASv = SAv = Sz$. Therefore, z is a common fixed point of A, B, S and T. Similarly we can also complete the proof when B is continuous.

In order to prove the uniqueness of the point z, suppose that z and w $(z \neq w)$ are common fixed points of A, B, S and T. Then by (3.2) we have

$$\begin{aligned} d^p(z,w) &= d^p(Az,Bw) \\ &\leq \phi(d^p(Az,Sz), d^p(Bw,Tw), \\ &\qquad d^p(Sz,Tw), d^p(Az,Tw), d^p(Bw,Sz)) \\ &= \phi(0,0,d^p(z,w),d^p(z,w),d^p(z,w)) < d^p(z,w), \end{aligned}$$

which is a contradiction. Therefore $z = w$. This completes the proof.

THEOREM 3.5. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions* (3.1), (3.2), (3.10) *and* (3.11), *where ϕ satisfies* (i) *and* (iii):

(iii) *for each $t > 0$,*

$$\max\{\phi(t,t,t,t,t),\ \phi(t,t,t,2^p t,0),\ \phi(t,t,t,0,2^p t)\} < t.$$

Then A, B, S and T have a unique common fixed point in X.

REMARK 3.1. In Theorems 3.4 and 3.5, if we put $p = 1$, we have the results of S. M. Kang, Y. J. Cho and G. Jungck [9].

Finally, we show the existence of the common fixed point for compatible mappings in Theorems 3.4 and 3.5.

EXAMPLE 3.1. Let $X = [0,1]$ with the Euclidean metric d. Define A, B, S and T by

$$Ax = \frac{1}{4}x^{1/2}, \quad Bx = \frac{1}{8}x^{1/2}, \quad Sx = x^{1/2}, \quad Tx = \frac{1}{2}x^{1/2}$$

for all x in X, respectively. Then $A(X) = [0, \frac{1}{4}] \subset [0, \frac{1}{2}] = T(X)$. Similarly, $B(X) \subset S(X)$. Moreover, it is easy to show that the pairs A, S and B, T are compatible.

Consider a function $\phi : (R^+)^5 \to R$ defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) = h \max\{t_1, t_2, t_3, t_4, t_5\}$$

for all t_1, t_2, t_3, t_4 and t_5 in R^+, where $\left(\frac{1}{4}\right)^p \leq h < \left(\frac{1}{2}\right)^p$. Then ϕ satisfies the condition (Φ) and the condition (iii). Furthermore, we obtain

$$\begin{aligned} d^p(Ax, By) &= \left(\frac{1}{4}\right)^p d^p(Sx, Ty) \\ &\leq \phi(d^p(Ax, Sx), d^p(By, Ty), \\ &\qquad d^p(Sx, Ty), d^p(Ax, Ty), d^p(By, Sx)) \end{aligned}$$

for all x, y in X. Thus, all the hypotheses of Theorems 3.4 and 3.5 are satisfied. Further, zero is a unique common fixed point of A, B, S and T.

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